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Measuring the FSR-inclusive $\pi^+\pi^-$ cross section

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Abstract. Final state radiation (FSR) in pion pair production cannot be calculated reliably because of the composite structure of the pions. However, FSR corrections have to be taken into account for a precise evaluation of the hadronic contribution to $g-2$ of the muon. The role of FSR in both energy scan and radiative return experiments is discussed. It is shown how FSR influences the pion form factor extraction from experimental data and, as a consequence, the evaluation of $a_\mu^{\rm had}$. In fact the $O(\alpha)$ FSR corrections should be included to reach the precision we are aiming at. We argue that for an extraction of the desired FSR-inclusive cross section $\sigma_{\rm had}^{(\gamma)}$ a photon-inclusive scan measurement of the "$e^+e^- \to \pi^+\pi^- + \text{photons}$" cross section is needed. For exclusive scan and radiative return measurements in contrast we have to rely on ad hoc FSR models if we want to obtain either $\sigma_{\rm had}^{(\gamma)}$ or the FSR-exclusive cross section $\sigma_{\rm had}^{(0)}$. We thus advocate to consider seriously precise photon-inclusive energy scan measurements at present and future low energy e^+e^--facilities. Then together with radiative return measurements from DAΦNE and BABAR and forthcoming scan measurements at VEPP-2000 we have a good chance to substantially improve the evaluation of $a_\mu^{\rm had}$ in the future.

1 Introduction

Photon vacuum polarization effects are sizable and therefore play an important role in electroweak precision physics. Because of the strong interactions between quarks and gluons the contributions of the low energy hadrons cannot be calculated by perturbative QCD. However, they may be obtained via a dispersion integral over the experimental e^+e^- annihilation data. A precise evaluation of hadronic effects in quantities like the running fine structure constant $\alpha(s)$ and of the muon anomalous magnetic moment a_μ thus depends directly on the precision of low energy "$e^+e^- \to \text{hadrons}$" cross sections $\sigma_{\rm had}$ [1–3]. Further theoretical efforts may help to some extent to reduce the theoretical uncertainties of these quantities [4]. However, new measurements of $\sigma_{\rm had}$ are indispensable for achieving substantial progress. Indeed, remarkable improvements have been achieved in recent years by the CMD-2 Collaboration [5] at Novosibirsk and the BES-II Collaboration [6] at Beijing. New results are expected soon from radiative return experiments by KLOE [7,8] at the Φ-factory DAΦNE at Frascati and from the B-factory at SLAC with BABAR [9].

The muon $g-2$ experiment at Brookhaven now has reached the level of 0.7 ppm in precision [10,11] for a measurement of a_μ and depending on which evaluation of $a_\mu^{\rm had}$ is adopted [3] reveals a deviation from the standard model prediction which could be as large as 3 standard deviations. Since the main source of uncertainty of the SM prediction arises from the hadronic contributions, a careful reconsideration of the determination of $a_\mu^{\rm had}$ is mandatory. In fact existing low energy "$e^+e^- \to \pi^+\pi^-$" data are inconsistent with the corresponding $I=1$ part obtained via CVC (conserved iso-vector current) from hadronic τ-decay spectra [3,12]. This is a problem which most likely can only be resolved by new experiments. Needless to say that the experimental inconsistencies also reduce our possibilities to obtain a more precise determination of $a_\mu^{\rm had}$ and hence to draw conclusions about possible "new physics" which also would contribute to a_μ.

Experiments that measure $\sigma_{\rm had}$ do this either via an energy scan ($2m_\pi \le s^{1/2} \le E_{\max}$) or they measure the invariant mass distribution of the hadronic final states $\mathrm{d}\sigma_{\rm had}/\mathrm{d}s'$ ($s' \le s$) at meson factories running at fixed s, using the radiative return due to the emission of hard initial state photons. From $\mathrm{d}\sigma_{\rm had}/\mathrm{d}s'$ the cross section $\sigma_{\rm had}(s')$ can here be extracted by factoring out the photon radiation[1].

[1] As has been pointed out in [13] already, the radiative return "mechanism" at leading order has the nice property that the usual convolution integral, relating the observed cross section (which includes photon radiation effects) to the physical cross section of actual interest, appears de-convoluted (photon radiation acts as a spectral analyzer) such that instead of factorization under convolution integrals one has point by point factorization. Higher order effects which give rise to multiple convolution integrals of course spoil this simple picture since by taking one derivative we get rid of one integration only

At increasing precision it becomes more and more difficult and challenging to extract the relevant "pseudo-observable" quantities with adequate precision. By "pseudo-observable" one understands quantities obtained from raw experimental data only via some theoretical input. For example, one has to unfold the raw data from photon radiation effects, where the initial state radiation (ISR) is universal to all e^+e^- annihilation processes, while the final state radiation (FSR) and the initial–final state interference (IFS) are process specific. The "pseudo-observable" we are interested in is the "hadronic blob" which corresponds to the imaginary part of the correlator of two hadronic electromagnetic currents: the one-photon irreducible contributions to the photon vacuum polarization (see [13] and references therein). Here we concentrate on low energy π pair production, a relatively simple hadron production channel which is dominating the hadronic contribution to the muon $g-2$.

For both the scan and the radiative return method we are facing three major sources of uncertainty affecting the extraction of $\sigma_{\pi\pi}$: the experimental error, the theoretical error due to neglecting higher order QED corrections and finally the uncertainty related to non-perturbative effects related to photon radiation from the final hadronic state. Currently, great efforts are made to reduce the experimental error below the 1 per cent level [5, 7–9]. QED corrections concerning low energy pion pair production have been considered e.g. in [13–16]. In the present article we will focus on the last of the mentioned error sources, the model error related to photons radiated from the final hadronic state. Here the problem is that the radiation of photons by the pions is poorly understood theoretically. Since perturbative QCD breaks down at low energies it is not possible to treat the final state pions in terms of their constituent quarks. On the other hand hard photons participating in the scattering process do probe the pion sub-structure. Treating pions as point-like scalar particles by simply applying scalar QED (sQED) is therefore also not a solution to the problem. What makes things even more complicated is the fact that we have to deal with non-perturbative QCD effects like intermediate ρ or ω resonances and photon radiation from such a hadronic state cannot be treated in a straightforward way. However, this contribution of real photon emission can be expected to be less important than the radiation from the final state pions. This is because the net charge of the intermediate hadronic state is zero and in addition the de Broglie wavelength of the dominant ρ and ω resonances is relatively small in respect to the typical wavelength of the radiated photons. Only sufficiently hard photons are able to probe the sub-structure of a hadronic composite state. The pions on the other hand are charged and have a much longer de Broglie wavelength. Unfortunately a similar argument does not help for the virtual corrections since virtual hard photons are always included and also cannot be eliminated by cutting out the "trouble-making" part of the phase space as it is possible for real photons. As a consequence their magnitude is not known and they cannot be subtracted from the hadronic final state without relying on specific models like sQED.

As mentioned above the quantity of interest is the correlator of the hadronic component of two electromagnetic currents including strong as well as electroweak corrections. From the latter only the photonic corrections are sizable. They correspond to the FSR correction in the hadron production processes [17, 18]. Since these corrections cannot be calculated reliably, the aim is to measure, if possible, the hadronic cross section $\sigma^{(\gamma)}_{\rm had}$ that is dressed by final state photons. We would obtain in this way directly the quantity to be inserted into the dispersion integrals for the hadronic contribution to the running fine structure constant $\alpha(s)$ and to the muon anomalous magnetic moment a_μ, respectively, at the next to leading level of accuracy.

Including FSR means to include photonic corrections to the irreducible hadronic photon self-energy. We thus address the question whether we can circumvent the FSR problem by performing an inclusive measurement, i.e., undress the data from ISR only. The question has been discussed already in a previous paper for the radiative return scenario [13]. The result was that in this case an inclusive measurement does not yield the quantity of interest at sufficient precision. In other words, without substantial loss of precision one cannot avoid the necessity to undress from all photon radiation, including the complete treatment of FSR. Lacking a precise theoretical understanding of photon radiation by hadrons, however, one has to rely on model assumptions like "generalized sQED" for the pions (treating pions as point-like modulo a form factor) to extract the undressed (FSR-exclusive) cross section $\sigma^{(0)}_{\rm had}(s)$ from the data in a first step. In order to obtain the FSR-inclusive cross section $\sigma^{(\gamma)}_{\rm had}(s)$ one has to add the appropriate FSR contribution "by hand" at the end.

On the other hand for completely inclusive scan measurements we can use the fact that ISR and FSR factorize to "subtract" ISR from the observed inclusive total cross section $\sigma_{\rm obs}$, leaving, up to $O(\alpha^2)$ IFS contributions, the desired FSR-inclusive cross section $\sigma^{(\gamma)}_{\rm had}(s)$. As we will see for pion pair production such a "subtraction" of ISR is also possible with excellent precision for realistic cuts on the pion angles provided they are chosen such that they break the ISR $\otimes$ FSR factorization only slightly. The inclusive measurement requires a high quality detector with high acceptance and good separation of π^0 versus γ ($\pi^+\pi^-\pi^0$ background).

At CMD-2 [5] so far a different strategy has been used which we will call the exclusive scan measurement. Here an event selection is applied such that only soft real photons are included which then can be corrected away. While there are no problems with real hard photons in this case one still has the problem that the virtual photon contributions from the loops include hard photon effects. The virtual contributions must be subtracted and one has to apply the Bloch–Nordsieck construction as well in order to get an infrared finite cross section. Because the effective theory applied (generalized sQED) is renormalizable, one obtains infrared (IR) and ultraviolet (UV) finite results.

Note that existing data at present do not allow us to determine $\sigma^{(\gamma)}_{\rm had}(s)$ as required for a precise determination of its contribution to $a_\mu^{\rm had}$. Modeling FSR by sQED we may estimate the size of the effect we have in mind: it is given at leading perturbative order by $\delta^\gamma a_\mu^{\rm had} = (38.6 \pm 1.0)\times 10^{-11}$. This has to be confronted with the final precision $\delta^{\rm exp} a_\mu^{\rm had} \sim 40\times 10^{-11}$ expected from the Brookhaven muon $g-2$ experiment.

Before settling this 1σ (in terms of the expected final experimental precision) effect, it is urgent to clarify the origin of the 3 times larger discrepancy between $e^+e^- \to \pi^+\pi^-$ data and the corresponding data obtained via CVC from τ spectral functions (in the energy range just above the ρ resonance), and the present unclear status of the ρ mass and width [19]. This issue can certainly be settled by the radiative return experiments with KLOE [8] at LNF/Frascati and with BABAR [9] at SLAC[2]. However, a new energy scan experiment, which is anyway mandatory for a clean measurement of the FSR-inclusive cross section, could also help to clarify the origin of the observed deviations. In addition, as we shall argue below, in radiative return measurements in order to get rid of the model-dependent FSR contribution, at least one of the following conditions has to be fulfilled:

(i) $\sigma_{\pi\pi}(s' < s) \gg \sigma_{\pi\pi}(s)$ (true especially for the ρ resonance region);
(ii) $s' \simeq s$ (soft photon region);
(iii) suppression of FSR by kinematic cuts. As we will see, at ϕ-factories model dependence becomes an insurmountable problem at low energies below about 500 MeV where we have to deal with large contributions of hard FSR photons which cannot be suppressed by cuts.

The above remarks together with the results presented in this paper strongly suggest that it would be desirable to revitalize the idea to perform an energy scan at the DAΦNE machine at Frascati [20] in a second step after running as a Φ-factory.

In the next section we discuss the model error of pion form factor extraction connected to the radiation of photons from hadronic final states in inclusive and exclusive scan scenarios. In addition we analyze to what extent kinematic cuts on the pion angles change the picture and, for the inclusive scan scenario, consider the impact of the model uncertainty on the determination of $a_\mu^{\rm had}$. Section 3 is devoted to the model uncertainty of extracting the pion form factor $|F_\pi(s')|^2$ for fixed s in radiative return experiments. We also address the question if the FSR contribution and its related model error can be estimated from a measurement of the pion forward–backward asymmetry $A_{\rm FB}$. In Appendix A we derive a general, model-independent formula for the inclusive cross section $\sigma(e^+e^- \to \gamma^* \to X + \text{photons})$, where ISR and FSR are treated in a factorized form, X being an arbitrary non-photonic final state. In Appendix B some of the used formulas connected to FSR within sQED or fermionic QED (fQED) are collected.

[2] We should mention that another very problematic energy region exists where experimental data are very poor or even controversial and which is important for the precise evaluation of $a_\mu^{\rm had}$: the energy range 1.4 to 2.0 GeV (between the upper limit of the VEPP-2M machine at Novosibirsk and the lower limit of the BEPC machine at Beijing). Radiative return measurements with BABAR and results expected from VEPP-2000 (upgraded VEPP-2M) will substantially improve results in this range

2 Model errors for inclusive and exclusive measurements in scan experiments

2.1 Inclusive scenario

We first present a case study of "$e^+e^- \to \pi^+\pi^- + n\gamma$" ($n = 0, 1, 2, \ldots$). Experimentally on an event by event basis it is not possible to distinguish a final state from an initial state photon. In an inclusive measurement events with any number of (initial and/or final state) photons are counted. The major question will be to what extent and at what accuracy we may evaluate FSR-inclusive cross sections from the experimental data. We first consider the measurement of the FSR-inclusive cross section[3] $\sigma^{(\gamma)}(s)$ in energy scan experiments.

Suppose for the moment that we would be able to calculate photon radiation from pions. Then we would have two possibilities:

(1) determine the undressed cross section $\sigma^{(0)}(s)$ by unfolding the observed cross section from all photon radiation and add the FSR as calculated by perturbation theory with desired accuracy, which yields $\sigma^{(\gamma)}$;
(2) determine an FSR-inclusive cross section by unfolding only the calculated ISR from the observed cross section, which yields $\hat\sigma^{(\gamma)}(s)$.

The question then is: to what accuracy does $\hat\sigma^{(\gamma)}(s)$ approximate $\sigma^{(\gamma)}(s)$? Since, actually, we do not know how to calculate photon radiation from pions in a model-independent way only the second approach is able to give a model-independent answer; however, then we do not know how well $\hat\sigma^{(\gamma)}(s)$ approximates the quantity of interest $\sigma^{(\gamma)}(s)$. What we will do in this case is make a "guesstimate" of the quality of the approximation by modeling FSR by generalized sQED.

For the error estimate the following factorization theorem is crucial: Neglecting the IFS contribution, being of $O(\alpha^2)$ due to charge conjugation invariance, the inclusive total cross section $\sigma_{\rm obs}(s)$ may be written in a factorized form,

$$\sigma_{\rm obs}(s) = \int {\rm d}s_{\rm V}\, \sigma^{(\gamma)}(s_{\rm V}) \rho_{\rm ini}^{\rm incl}(s, s_{\rm V}) + O(\alpha^2)_{\rm IFS}, \tag{2.1}$$

which means that FSR and ISR can be treated independently from each other. Details are given in Appendix A

[3] In the following we drop the labels "had" or "$\pi\pi$" for cross sections like $\sigma^{(0)}$ and $\sigma^{(\gamma)}$

[see (A.21) and (A.28)]. It is worth to stress that the powerful identity (2.1) is not easily recognizable to fixed perturbative order. Note that (2.1) is quite general once IFS is neglected. It is based on the fact that we have a neutral current process for which we can apply a separation into Lorentz covariant and individually gauge invariant initial and final state contributions. Qualitatively the result may be understood as follows: by the fact that at low energies the single virtual photon exchange ($1/s$-enhancement) highly dominates the "$e^+e^- \to \pi^+\pi^- + n\gamma$" cross section and due to the suppression of the IFS (see below) it makes sense to consider the process in an approximation of an s-channel single photon exchange (i.e. diagrams which factorize into two disconnected parts upon cutting the photon line). This virtual photon then carries the invariant mass $s_{\rm V}^{1/2}$ and the above convolution is exact up to the indicated missing IFS effects. We would like to remind the reader that the representation of the $\pi\pi$-production cross section in terms of the pion form factor[4]

$$\sigma^{(0)}(s) = |F_\pi^{(0)}(s)|^2 \sigma^{0,\rm point}(s), \tag{2.2}$$

with [$\beta_\pi = (1-4m_\pi^2/s)^{1/2}$ is the pion velocity]

$$\sigma^{0,\rm point}(s) = \frac{\pi}{3}\frac{\alpha^2\beta_\pi^3}{s},$$

also makes sense strictly only for the one-photon exchange approximation[5]. In this approximation $s_{\rm V}$ can be neatly identified with s in $|F_\pi(s)|^2$, in spite of the fact that $s_{\rm V}$, as the squared invariant mass of a virtual state, is not an observable. Thus in (2.1) $s_{\rm V}$ is only a formal (unphysical) integration variable where the boundaries are physical observables: $4m_\pi^2 \le s_{\rm V} \le s$. Nevertheless, we can always fit the pseudo-observable $\sigma^{(\gamma)}(s_{\rm V})$ to the observed data $\sigma_{\rm obs}(s)$ by using (2.1) and thereby determine $\sigma^{(\gamma)}(s_{\rm V})$. The accuracy with which this can be achieved, up to IFS contributions, is limited by our knowledge of the initial state radiator function $\rho_{\rm ini}^{\rm incl}(s, s_{\rm V})$ only. The latter can be calculated without any model dependence within perturbative QED (see e.g. [21]). Since IFS effects are of $O(\alpha^2)$, the model dependence for the extraction of the FSR-inclusive cross section is determined by an (as yet unknown) $O(\alpha^2)$ IFS contribution[6]. What is very important is that IFS interference does not include contributions from leading logarithms of the kind $\log(s/m_e^2)$. We may estimate the $O(\alpha^2)$ effect to be at the per mill level.

[4] Note that $\sigma^{(0)}(s)$ and equivalently $|F_\pi^{(0)}(s)|^2$ are not measurable quantities, as we shall discuss below. They are useful, theoretically motivated concepts defined in a world where the electroweak interactions are switched off. In reality we cannot switch off QED effects and this is part of the problem we are dealing with in this paper. If one could calculate $|F_\pi^{(0)}(s)|^2$ for time-like s non-perturbatively in lattice QCD, this is the quantity what one would take from lattice QCD

[5] Since in (2.2) $F_\pi^{(0)}(s)$ does not depend on the pion production angle a similar formula (B.18) is valid for the case of angular cuts with the same function $F_\pi^{(0)}(s)$

[6] The $O(\alpha^2)$ IFS is complete for the real photon emission. However, some virtual contributions are not yet calculated

As we have already stressed, our "master formula" (2.1) cannot be directly applied to a real experiment with some cuts and/or detector inefficiencies (the leading uncertainties are due to the need of extrapolation to the blind zones of the measurement). In a real experiment the influence of these effects will be taken into account using a realistic Monte Carlo event generator which features a high quality ISR matrix element and some modeling of FSR. Let us focus therefore on a situation where angular cuts are present. Then, ISR and FSR phase space integrations cannot be disentangled as it is possible without cuts [see (A.11)] and ISR $\otimes$ FSR factorization breaks down at the $O(\alpha)$ level. As a consequence, to subtract only ISR from the data we have to rely on specific FSR models. To be able to extract $\sigma^{(\gamma)}(s_{\rm V})$ without significant model dependence the condition that *the applied cuts break* ISR$\otimes$FSR *factorization only slightly* has to be fulfilled. Here we will investigate the breaking of ISR $\otimes$ FSR factorization by some semi-realistic C-symmetric cuts, $\Theta_{\pi_\pm} \ge \Theta_\pi^M$, $\Theta_{\pi_\pm}$ being the laboratory angle between the pion momenta and the beam axis, treating FSR by sQED. For such cuts we then can write the observed cross section as (for details see Appendix B and [13]; Λ is the soft photon energy separating soft from hard photons)

$$\begin{aligned}\sigma_{\rm obs}^{\rm cut}(s) &= \sigma_{\rm cut}^{(\gamma)}(s)\left[1+\delta_{\rm ini}(s,\Lambda)\right] \\ &\quad + \int_{4m_\pi^2}^{s-2\sqrt{s}\Lambda} {\rm d}s_{\rm V}\,\sigma_{\rm cut}^{(\gamma)}(s_{\rm V})\rho_{\rm ini}^{\rm cut}(s,s_{\rm V}) - \delta_{\rm cut}^{\rm scan}(s),\end{aligned} \tag{2.3}$$

with $\delta_{\rm ini}(s,\Lambda)$ corresponding to the soft plus virtual and $\rho_{\rm ini}^{\rm cut}(s,s_{\rm V})$ corresponding to hard photon initial state QED corrections. The $O(\alpha)$ ISR $\otimes$ FSR factorization breaking term $\delta_{\rm cut}^{\rm scan}(s)$ accounts for the missing pion events which cannot be seen in the experiment:

$$\delta_{\rm cut}^{\rm scan}(s) = \sigma_{\rm cut}^{(0)}(s)\frac{\alpha}{\pi}\left\{\eta(s)-\eta_{\rm cut}(s)\right\} + O(\alpha^2). \tag{2.4}$$

$\delta_{\rm cut}^{\rm scan}$ vanishes for the case without cuts (restoration of factorization). Remember that for C-symmetric angular cuts the $O(\alpha)$ IFS contribution drops out.

In a world with point-like pions we could calculate $\delta_{\rm cut}^{\rm scan}(s)$ perturbatively in sQED, where

$$\eta_{(\rm cut)}(s) = \frac{\pi}{\alpha}\left[\delta_{\rm fin}(s,\Lambda) + \int_{4m_\pi^2}^{s-2\sqrt{s}\Lambda} {\rm d}s'\rho_{\rm fin}^{(\rm cut)}(s,s')\right], \tag{2.5}$$

with s' being the square of the invariant mass of the pion pair and $\delta_{\rm fin}(s,\Lambda)$ and $\rho_{\rm fin}^{(\rm cut)}(s,s')$ the corresponding FSR corrections given in Appendix B. For real world pions we may estimate this term assuming generalized sQED which at least treats the soft photon part correctly and for the rest is a guess. It means that we assume that (2.4) and (2.5), with $\eta(s)$ calculated in sQED, still to some approximation account for the effect. What we will actually do is to consider $\delta_{\rm cut}^{\rm scan}(s)$, evaluated as just described, as a theoretical uncertainty (model error).

Note that (2.3) only contains the measured cross section $\sigma_{\rm obs}^{\rm cut}(s)$, the known initial state correction factors $\delta_{\rm ini}$

and $\rho_{\rm ini}^{\rm (cut)}$, and the FSR-inclusive cross section $\sigma_{\rm cut}^{(\gamma)}(s)$, which is the quantity to be extracted from the data since it corresponds to the FSR-inclusive pion form factor via (B.20).

Whether the approximation $\hat{\sigma}_{\rm cut}^{(\gamma)}(s)$ obtained via

$$\sigma_{\rm obs}^{\rm cut}(s) = \hat{\sigma}_{\rm cut}^{(\gamma)}(s)\,[1+\delta_{\rm ini}(s,\Lambda)] + \int_{4m_\pi^2}^{s-2\sqrt{s}\Lambda} {\rm d}s'\hat{\sigma}_{\rm cut}^{(\gamma)}(s')\rho_{\rm ini}^{\rm cut}(s,s'), \qquad (2.6)$$

after neglecting $\delta_{\rm cut}^{\rm scan}(s)$ in (2.3), yields a good approximation for the FSR-inclusive cross section is subject of the investigation described in the following.

We first introduce the FSR-inclusive form factor $F_\pi^{(\gamma)}(s)$ by

$$|F_\pi^{(\gamma)}(s)|^2 = |F_\pi^{(0)}(s)|^2\left(1+\frac{\alpha}{\pi}\eta(s)\right) + O(\alpha^2), \qquad (2.7)$$

and *assume* that in some approximation it makes sense to write formulas like (2.2) also between $\sigma^{(\gamma)}(s)$ and $F_\pi^{(\gamma)}(s)$ and between $\hat{\sigma}^{(\gamma)}(s)$ and $\hat{F}_\pi^{(\gamma)}(s)$. Hard photon effects spoil these assumptions at some level, but this at the moment is difficult to quantify. So in the following, this will be part of our model assumption (see below).

To estimate the model dependence for the extraction of $\sigma_{\rm cut}^{(\gamma)}(s)$ we first generate a sample $\sigma_{\rm obs}^{\rm cut}(s)$, using the pion form factor $|F_\pi^{(0)}(s)|^2$ as given in [5] and the relations (2.3)–(2.5) and (B.18)–(B.21). Then we utilize the MINUIT package [22] to obtain $|\hat{F}_\pi^{(\gamma)}(s)|^2$ [which corresponds to $\hat{\sigma}_{\rm cut}^{(\gamma)}(s)$] from the $\sigma_{\rm obs}^{\rm cut}(s)$ data using (2.6). For the data fitting we adopt again the Gounaris–Sakurai type parameterization of $|F_\pi^{(0)}(s)|^2$ in the version proposed in [5].

We then estimate the model error by

$$\Delta_{\rm cut}^{\rm scan}(s) = \frac{\sigma_{\rm cut}^{(\gamma)}(s)-\hat{\sigma}_{\rm cut}^{(\gamma)}(s)}{\sigma_{\rm cut}^{(\gamma)}(s)} = \frac{|F_\pi^{(\gamma)}(s)|^2-|\hat{F}_\pi^{(\gamma)}(s)|^2}{|F_\pi^{(\gamma)}(s)|^2}. \qquad (2.8)$$

The model error for the extraction of $|F_\pi^{(\gamma)}(s)|^2$ from scan data with C-symmetric cuts is shown in Fig. 1. The detailed shape of the curve depends on the parameterization of the pion form factor. It can be noticed that the considered cuts do not lead to large model errors. Even for the extreme cut of $\Theta_\pi \geq 60°$ the model error is still below half a per cent, for $\Theta_\pi \geq 30°$ it is below 1 per mill. In fact, the observed smallness of the model error is related to the p-wave-like angular distribution of the outgoing pions. Figure 2 shows that angular cuts of the pion angle against the beam axis up to 30° decrease the cross section very little.

In Fig. 3 the impact of the discussed model error on $a_\mu^{\rm had}$,

$$a_\mu^{\rm had} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2\int_{4m_\pi^2}^{\infty}{\rm d}s\frac{R_{\pi\pi}(s)\hat{K}(s)}{s^2}, \qquad (2.9)$$

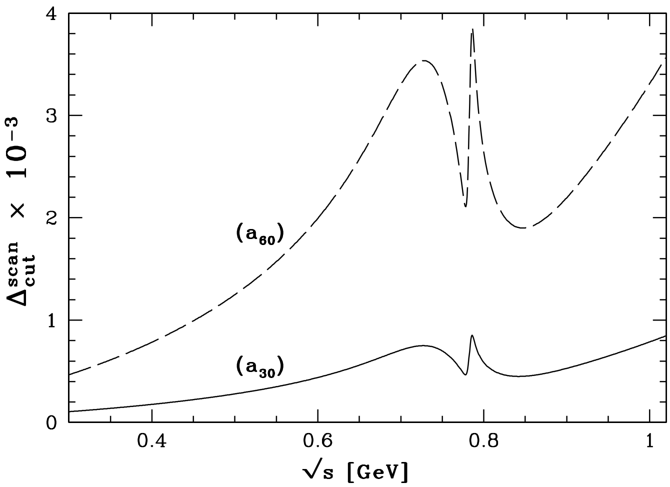

Fig. 1. Estimated relative model error for the extraction of the absolute square of the FSR-inclusive pion form factor, $|F_\pi^{(\gamma)}(s)|^2$, as a function of the center-of-mass energy in a photon-inclusive scan experiments for different C-symmetric angular cuts on the pion angles. Curve (a$_{30}$) and curve (a$_{60}$) correspond to the cuts $\Theta_\pi \geq 30°$ and $\Theta_\pi \geq 60°$, respectively

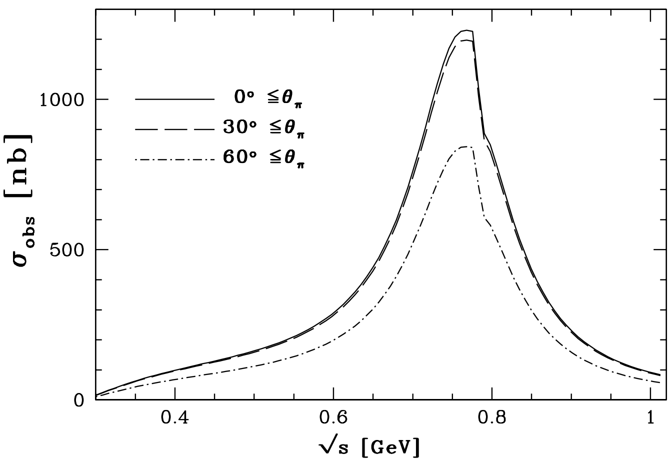

Fig. 2. Observed total cross section for pion pair production as a function of the center-of-mass energy ($s^{1/2} \leq 1.02\,$GeV) for the considered cut scenarios

is shown. For this we compare the values of $a_\mu^{\rm had}$ when inserting

$$R_{\pi\pi}(s) \equiv R_{\pi\pi}^{(\gamma)}(s) = \frac{\beta_\pi^3}{4}|F_\pi^{(\gamma)}(s)|^2 \qquad (2.10)$$

into (2.9) [this value is denoted by $a_\mu^{{\rm had}(\gamma)}$] with the value when inserting

$$R_{\pi\pi}(s) \equiv \hat{R}_{\pi\pi}^{(\gamma)}(s) = \frac{\beta_\pi^3}{4}|\hat{F}_\pi^{(\gamma)}(s)|^2 \qquad (2.11)$$

[this value is denoted by $\hat{a}_\mu^{{\rm had}(\gamma)}$]. Then the model error of $a_\mu^{\rm had}$ which is plotted in Fig. 3 for the curves (a_{30}) and (a_{60}) is defined as

$$\Delta a_\mu^{\rm had} = \frac{a_\mu^{{\rm had}(\gamma)}-\hat{a}_\mu^{{\rm had}(\gamma)}}{a_\mu^{{\rm had}(\gamma)}}. \qquad (2.12)$$

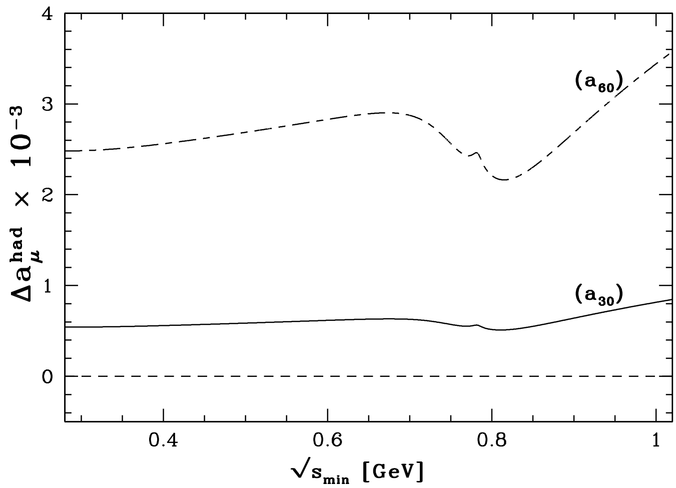

Fig. 3. Estimated relative model error in per mill of $a_\mu^{\rm had}$ ($s_{\rm min} < s < M_\Phi^2$) in scan experiments. The curves (a_{30}) and (a_{60}) correspond to the cases in Fig. 1

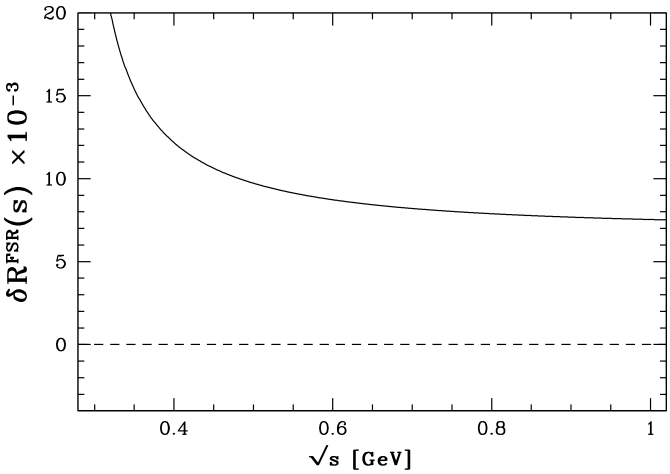

Fig. 4. Importance of FSR in sQED. The curve shows the difference between the absolute square of the undressed and of the FSR-inclusive pion form factor [see (2.13)]

Let us recall that the present theoretical error is at the level of 1.2% per cent.

As a first summary we may say that the direct extraction of $|\hat{F}_\pi^{(\gamma)}(s)|^2$ in an inclusive scan yields a very good approximation of $|F_\pi^{(\gamma)}(s)|^2$, especially in the low energy region where the contribution of FSR becomes large.

2.2 Exclusive scenario

Scan measurements in the past attempted to extract the "bare" cross section $\sigma^{(0)}(s)$, undressed from photon radiation effects. As already mentioned, the bare cross section is the object of primary theoretical interest, in principle. It is the quantity which allows us to extract the pion form factor which encodes the strong interaction structure of the pion in a world where the electromagnetic interaction has been switched off. It is the non-perturbative quantity which one would compute by a simulation in lattice QCD or investigate by means of general low energy properties of the strong interactions like chiral perturbation theory, locality and analyticity [4]. The theoretical concept of disentangling effects from different interactions has been very successful; however, it has its limitation at some point. In the phenomenology of low energy hadrons it is in fact not possible to separate in a model-independent way QED from QCD effects (at the level of accuracy we are considering here weak interaction effects are negligible). This will be discussed in more detail in the next section. Here, for the moment, we assume that $\sigma^{(0)}(s)$ is a sensible pseudo-observable.

Figure 4 shows the relative deviation of the FSR-inclusive pion form factor $|F_\pi^{(\gamma)}(s)|^2$ (being the desired quantity to be inserted into the dispersion integrals) from the undressed pion form factor $|F_\pi^{(0)}(s)|^2$, as calculated within sQED [see (B.14), (B.18) and (B.20)]

$$\delta R^{\rm FSR}(s) = \frac{\sigma^{(\gamma)}(s) - \sigma^{(0)}(s)}{\sigma^{(0)}(s)}$$

$$= \frac{|F_\pi^{(\gamma)}(s)|^2 - |F_\pi^{(0)}(s)|^2}{|F_\pi^{(0)}(s)|^2}$$
$$= \frac{\alpha}{\pi}\eta(s) + O(\alpha^2). \qquad (2.13)$$

This quantity can be taken as a measure of the importance of FSR.

Since the analysis presented so far does not account for the fact that generalized sQED describes correctly the soft photon part of the FSR-spectrum only[7], we would like to go further and compare sQED modeling with a second one which differs from it for hard photons. As in [13] we compare two different treatments of final state corrections: once we take for $\rho_{\rm fin}(s,s')$ and $\delta_{\rm fin}(s,\Lambda)$ the functions related to photonic radiation from point-like, scalar pions, and once we use the corresponding functions for the photonic radiation from point-like, fermionic pions with the same charge and mass (see Appendix B). The reason we do this is that in the soft photon limit the scalar as well as the fermionic approach yields the same correct result for the real photon contribution. For hard photons on the other hand both scenarios are obviously different. This will allow us to get some feeling in what kinematic regions hard photons play a substantial role and what model uncertainty we have to expect.

We thus consider in the following the model dependence of FSR effects, by replacing the integrated final state corrections for scalar particles, represented by the factor $\eta(s)$, by a corresponding factor for a fermionic final state, which we denote by $\eta^{\rm f}(s)$ (see again Appendix B for explicit formulas). Though both $\eta(s)$ and $\eta^{\rm f}(s)$ diverge when approaching the pion pair production threshold (Coulomb

[7] Note that for $s \to 4m_\pi^2$ real FSR is known precisely since there is only enough phase space for the radiation of soft photons, and for soft photons the FSR radiation mechanism is universal for given masses and charges of the final state particles. Therefore estimating the model uncertainty to be given by the sQED result appears to be too crude as it overestimates the model uncertainty in the soft photon region

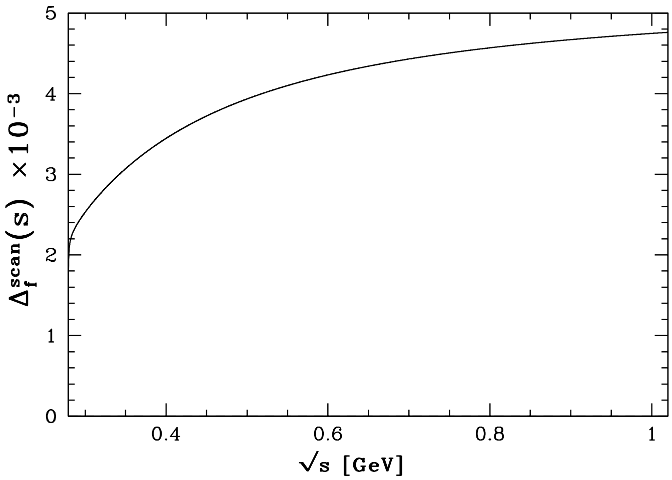

Fig. 5. Model estimate of the relative model uncertainty for the extraction of the absolute square of the FSR-inclusive pion form factor, $|F_\pi^{(\gamma)}(s)|^2$, as a function of the center-of-mass energy in scan experiments

pole[8]), their difference in this limit is a small number, $\lim_{s\to 4m_\pi^2}[\eta(s)-\eta^{\mathrm{f}}(s)] = 1$ (in units α/π)[9]. As already mentioned, obviously, the generalized sQED/fQED modeling of FSR obtained by replacing the point-pion form factor "1" by a form factor function $|F_\pi(s)|^2$ of one single variable s is valid for soft photons only. A method which will allow us to describe also hard photons in a realistic manner will be presented in a forthcoming paper.

Our discussion thus motivates the consideration of the following measure of the model dependence:

$$\Delta_{\mathrm{f}}^{\mathrm{scan}}(s) = \frac{\sigma^{(\gamma)}-\sigma^{(\gamma),\mathrm{f}}}{\sigma^{(\gamma)}} = \frac{|F_\pi^{(\gamma)}(s)|^2-|F_\pi^{(\gamma),\mathrm{f}}(s)|^2}{|F_\pi^{(\gamma)}(s)|^2} = \frac{\alpha}{\pi}[\eta(s)-\eta^{\mathrm{f}}(s)] + O(\alpha^2). \qquad (2.14)$$

It compares in a ratio $|F_\pi^{(\gamma)}(s)|^2$, being the pion form factor dressed by scalar FSR, with the absolute square of the pion form factor $|F_\pi^{(\gamma),\mathrm{f}}(s)|^2$, being dressed by fermionic FSR, and corresponds to our ignorance of FSR. Figure 5 shows $\Delta_{\mathrm{f}}^{\mathrm{scan}}(s)$ as a function of energy and suggests an uncertainty below 0.5% over the whole energy range of interest.

For the data analysis at the CMD-2 experiment [5] the event selection was such that only events containing real low energy photons were taken into account and thus the pions were approximately back to back. Since for soft photons the FSR mechanism is known (factorization), the real photonic corrections together with the universal soft plus virtual IR terms can be subtracted from the observed cross section in an essentially model-independent way[10]. Accordingly, in order to keep the formulas simple, we define a subtracted cross section $\sigma_{\mathrm{obs}}^{\mathrm{subtr}}(s)$ which does not depend on the cuts any longer and is obtained from the experimentally observed exclusive cross section in a theoretically well controlled manner.

While real hard photons may be eliminated by appropriate cuts, the same cannot be done for the remaining virtual corrections which include high momentum scales in loops and hence their treatment is model dependent. As a consequence the determination of the undressed cross section $\sigma^{(0)}(s)$ as well as of the FSR-inclusive cross section $\sigma^{(\gamma)}(s)$ from the given data suffers from model dependence which is hard to estimate.

Here we will present a model error estimate assuming that the observed cross section can be written as a product of $\sigma^{(0)}(s)$ containing all QCD effects (pion form factor) and a function containing only the initial and final state QED corrections. This ad hoc assumption, although criticizable, seems to be the best we can do so far. Applying the procedure of real and IR photon subtraction as described above then yields

$$\sigma_{\mathrm{obs}}^{\mathrm{subtr}}(s) \simeq \sigma^{(0)}(s)\left[1+\tilde\delta_{\mathrm{ini}}^{\mathrm{V+S}}(s)+\tilde\delta_{\mathrm{fin}}^{\mathrm{V+S}}(s)\right], \qquad (2.15)$$

which is the $O(\alpha)$ cross section including only the non-IR initial and final state soft plus virtual corrections corresponding to $\tilde\delta_{\mathrm{ini}}^{\mathrm{V+S}}(s)$ and $\tilde\delta_{\mathrm{fin}}^{\mathrm{V+S}}(s)$. Here of course the final state correction factor $\tilde\delta_{\mathrm{fin}}^{\mathrm{V+S}}(s)$ is not known. Thus at this stage we have two unknowns: $\sigma^{(0)}(s)$ and the FSR correction. Only after assuming that FSR is given by sQED or fQED we can then extract the undressed or the FSR-inclusive cross section from $\sigma_{\mathrm{obs}}^{\mathrm{subtr}}(s)$ via the following formulas (see Appendix B):

$$\hat\sigma^{(0)(,\mathrm{f})}(s) = \sigma_{\mathrm{obs}}^{\mathrm{subtr}}(s)\frac{1}{1+\tilde\delta_{\mathrm{ini}}^{\mathrm{V+S}}(s)+\tilde\delta_{\mathrm{fin}(,\mathrm{f})}^{\mathrm{V+S}}(s)}, \qquad (2.16)$$

$$\hat\sigma^{(\gamma)(,\mathrm{f})}(s) = \sigma_{\mathrm{obs}}^{\mathrm{subtr}}(s)\frac{1+\frac{\alpha}{\pi}\eta^{(\mathrm{f})}(s)}{1+\tilde\delta_{\mathrm{ini}}^{\mathrm{V+S}}(s)+\tilde\delta_{\mathrm{fin}(,\mathrm{f})}^{\mathrm{V+S}}(s)}. \qquad (2.17)$$

Using (2.16) we could try to estimate the uncertainty for the extraction of $\hat\sigma^{(0)}(s)$ via

$$\Delta_{\mathrm{scan},0}^{\mathrm{excl.}}(s) = \frac{\hat\sigma^{(0)}(s)-\hat\sigma^{(0),\mathrm{f}}(s)}{\hat\sigma^{(0)}(s)} = 1-\frac{1+\tilde\delta_{\mathrm{ini}}^{\mathrm{V+S}}(s)+\tilde\delta_{\mathrm{fin}}^{\mathrm{V+S}}(s)}{1+\tilde\delta_{\mathrm{ini}}^{\mathrm{V+S}}(s)+\tilde\delta_{\mathrm{fin}}^{\mathrm{V+S,f}}(s)}. \qquad (2.18)$$

The such estimated model error is shown in Fig. 6 [curve (0)]. We would like to stress that we should be careful not to take this error estimate obtained from the comparison of two factorizable models too seriously. In fact, if we would make the analogous comparison for the extraction of $\hat\sigma^{(\gamma)}(s)$ the such obtained error would be of the level of 1 per mill. However, we would expect a larger error from non-factorizable FSR contributions which cannot be estimated.

[8] The Coulomb resummation has been considered in [13]

[9] At high energies the scalar and fermionic FSR read $\eta(s\to\infty)=3$ and $\eta^{\mathrm{f}}(s\to\infty)=3/4$

[10] Collinear hard photons can be easily separated via the event shapes

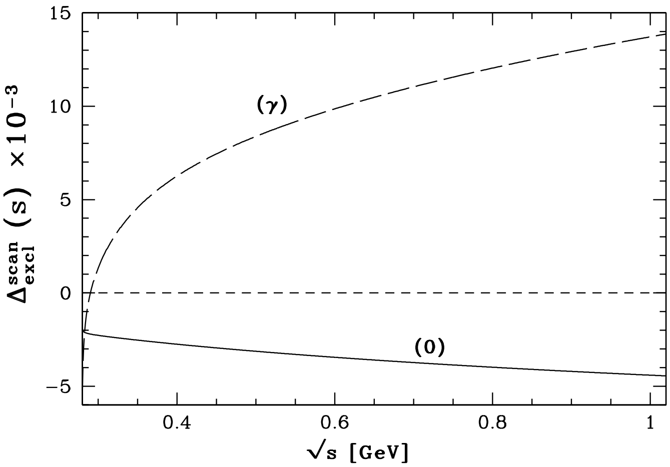

Fig. 6. Model error estimations for the extraction of the absolute square of the pion form factor in an exclusive scenario [see (2.18) and (2.20)]

As an alternative possibility we may try, in the spirit of (2.1), what we get if we just correct for the model-independent ISR and the model-independent IR-sensitive part of FSR, obtaining

$$\tilde{\sigma}^{(\gamma)}(s) = \frac{\sigma_{\rm obs}^{\rm subtr}(s)}{1+\tilde{\delta}_{\rm ini}^{\rm V+S}(s)}. \qquad (2.19)$$

Defining

$$\Delta_{{\rm scan},\gamma}^{\rm excl.}(s) = \frac{\tilde{\sigma}^{(\gamma)}(s) - \hat{\sigma}^{(\gamma)}(s)}{\tilde{\sigma}^{(\gamma)}(s)} = 1 - \frac{\left[1+\frac{\alpha}{\pi}\eta(s)\right]\left[1+\tilde{\delta}_{\rm ini}^{\rm V+S}(s)\right]}{1+\tilde{\delta}_{\rm ini}^{\rm V+S}(s)+\tilde{\delta}_{\rm fin}^{\rm V+S}(s)} \qquad (2.20)$$

we can get a feeling for how well $\tilde{\sigma}^{(\gamma)}(s)$ approximates the true $\sigma^{(\gamma)}(s)$ [estimated here by (2.17)]. The result is shown in Fig. 6 [curve (γ)].

To summarize: what can we get from a hard-photon exclusive measurement?

(i) $\sigma^{(0)}$: in spite of the fact that all real hard photons have been eliminated by cuts a surprisingly large model uncertainty due to hard virtual photons poses an inherent limitation: the corresponding uncertainty cannot fall below the level of about 0.5% (sQED). Strictly speaking $\sigma^{(0)}$ is not accessible to experiment or only at limited precision by the fact that we cannot switch off virtual QED effects in reality.

(ii) $\sigma^{(\gamma)}$: the missing real hard photons must be calculated from a model like sQED and added by hand. What we get is a model-dependent $\hat{\sigma}^{(\gamma),({\rm model})}(s)$. Surprisingly, the model dependence we estimate by our method (assuming factorization with a single scale form factor) for this object is much smaller (at the level of 0.1% only). On the one hand this reduced model dependence can be traced back to the Kinoshita–Lee–Nauenberg (KLN) theorem, which infers that radiative corrections for total inclusive cross sections are free from large logs. On the other hand it is not conceivable that we get a more precise knowledge of $\sigma^{(\gamma)}$ from not measuring hard photons than from actually measuring everything. In the latter case the uncertainty shown in Fig. 1 has been estimated, which, as expected, shows an increasing uncertainty for increasingly strong cuts. Nevertheless, even though we think that our method of estimating the model dependence underestimates the error in the exclusive case, it is a quantity which is protected by the KLN theorem from large effects and thus is a quantity which seems to be under much better control than e.g. the bare $\sigma^{(0)}$. How much better is hard to quantify at this stage.

(iii) $\tilde{\sigma}^{(\gamma)}(s)$: is model-independent per definition but it is not the quantity of actual interest, as it is not a good approximation to $\sigma^{(\gamma)}$. After all the hard real photons are missing here and again we only can get what we are interested in by adding the missing piece using a model. If we do so we end up with $\hat{\sigma}^{(\gamma),({\rm model})}(s)$ again, up to higher order terms. Then we are essentially back at (ii).

3 Model errors in radiative return measurements

At radiative return experiments the spectral function $d\sigma/ds'$ is measured where $s'^{1/2}$ is the invariant mass of the non-photonic final state. Let us first have a look at our "master formula" (2.1) which is the photon-inclusive cross section in form of a convolution integral in the integration variable $s_{\rm V}$, $s_{\rm V}$ being the invariant mass square of the hadronic final state including the FSR photons but excluding the ISR photons. One could think that, due to the factorization of ISR and FSR already on the matrix element level [see (A.5)], it is possible to extract $\sigma^{(\gamma)}(s_{\rm V})$ also from a radiative return measurement at fixed s. Of course by rewriting (2.1) we can formally get

$$\frac{d\sigma_{\rm incl}}{ds_{\rm V}} = \sigma^{(\gamma)}(s_{\rm V})\rho_{\rm ini}^{\rm incl}(s,s_{\rm V}) + O(\alpha^2)_{\rm IFS}. \qquad (3.1)$$

However, we cannot extract $\sigma^{(\gamma)}(s_{\rm V})$ from the experimental data for the simple reason that $s_{\rm V}$ is not an observable. This can be immediately seen already for the case of single photon emission where we have $s_{\rm V} = s'$ if the photon is emitted from the initial state, but $s_{\rm V} = s$ if the photon is emitted from the final state. Since on an event level ISR and FSR photons cannot be distinguished, $s_{\rm V}$ cannot be obtained from the observables s and s'. The error we would make when identifying s' with $s_{\rm V}$ is therefore of leading order FSR. This is one way to see that not only there is no way to measure the FSR-inclusive cross section $\sigma^{(\gamma)}(s)$ in a radiative return measurement, but, in fact, we have to deal with an $O(1)$ FSR background leading to an in general significant model error for the extraction of even the undressed cross section $\sigma^{(0)}(s)$. In the following we therefore are going to investigate the model error for radiative return scenarios in a separate analysis.

Taking into account radiative corrections in the approximation where only the leading single photon radiation from the final state is included, the observed spectral function can be expressed as the sum of an ISR and an FSR contribution since the IFS contribution drops out:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)_{\mathrm{obs}} = \sigma^{(0)}(s')\tilde{\rho}_{\mathrm{ini}}(s,s') + \sigma^{(0)}(s)\tilde{\rho}_{\mathrm{fin}}(s,s'). \quad (3.2)$$

Again we refer to Appendix B and [13] for the explicit expressions. Because we are interested in measuring the FSR-inclusive cross section it is tempting to extract the quantity

$$\tilde{\sigma}^{(\gamma)}(s') = \frac{1}{\tilde{\rho}_{\mathrm{ini}}(s,s')}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)_{\mathrm{obs}}, \quad (3.3)$$

which is obtained by just subtracting the ISR part and dropping the model-dependent last term of (3.2). However, since what we really want to get is $\sigma^{(\gamma)}(s)$, we have to rewrite (3.2) in terms of this true $O(\alpha)$ FSR-inclusive cross section. We easily find

$$\sigma^{(\gamma)}(s') = \frac{1}{\tilde{\rho}_{\mathrm{ini}}(s,s')}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)_{\mathrm{obs}} + \delta^{\mathrm{r.r.}}(s,s'), \quad (3.4)$$

with

$$\delta^{\mathrm{r.r.}}(s,s') = -\frac{\tilde{\rho}_{\mathrm{fin}}(s,s')}{\tilde{\rho}_{\mathrm{ini}}(s,s')}\sigma^{(0)}(s) + \frac{\alpha}{\pi}\eta(s')\sigma^{(0)}(s'). \quad (3.5)$$

We can see that the first term of (3.5) is of $O(1)$ but the second of $O(\alpha)$, so that no cancellation up to higher order terms is possible! The first term may be considered as a correction term only in the region where we have a large enhancement of $\sigma^{(0)}(s')$ by the ρ resonance, in particular, in comparison to the reference cross section $\sigma^{(0)}(s)$ at $s = M_{\Phi}^2$ (or M_B^2). In addition, the mass effects of photon radiation by the pions versus the ones from the electron–positron system in fact lead to quite some suppression of $\tilde{\rho}_{\mathrm{fin}}(s,s')$ in comparison to $\tilde{\rho}_{\mathrm{ini}}(s,s')$, both of which are of $O(\alpha)$. We conclude that, in radiative return experiments, a direct model-independent extraction of the FSR-inclusive $\sigma^{(\gamma)}(s')$ is not possible at $O(\alpha)$ precision in the naive way just considered.

Since we try here to discuss $O(\alpha)$ corrections to $\sigma^{(0)}(s')$, which in the radiative return scenario is given by

$$\sigma^{(0)}(s') = \frac{1}{\tilde{\rho}_{\mathrm{ini}}(s,s')}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)_{\mathrm{obs}} - \frac{\tilde{\rho}_{\mathrm{fin}}(s,s')}{\tilde{\rho}_{\mathrm{ini}}(s,s')}\sigma^{(0)}(s), \quad (3.6)$$

one obvious deficiency of our starting equation (3.2) is the missing higher order corrections. In the resolved form (3.6) we are at the $O(1)$ level only and thus we are not able to seriously address the question of an FSR-inclusive measurement. Obviously we lose one order in α in a radiative return measurement. A discussion beyond leading order FSR would be possible only if the complete next order version of (3.2) would be available. The need for going to higher orders also leads to more problems with the treatment of hard photons radiated by the pions. Of course the limitations are coming from the fact that virtual hard photon emission by the pions cannot be switched off and our limited theoretical knowledge of the higher order contributions has its drawbacks for a precise determination of $\sigma^{(0)}(s')$ or $\sigma^{(\gamma)}(s')$. The basic problems and limitations discussed above for exclusive scan measurements also apply for the radiative return method. Nevertheless, a discussion on the basis of (3.2) addresses the major difficulty we encounter in the attempt to measure the FSR-dressed cross section in a radiative return experiment.

As (3.4) tells us, the FSR correction $(\alpha/\pi)\eta(s')$, given precisely by the second term of (3.5), is *completely lost* once we drop $\delta^{\mathrm{r.r.}}(s,s')$ in order to get the model-independent quantity (3.3), which means that the latter is not a very meaningful quantity. Therefore, in [13] we proposed to unfold the raw data from all photon radiation, by modeling FSR by generalized sQED. The extracted undressed cross section $\sigma^{(0)}(s)$ then suffers from model dependence. From the preceding discussion we know what the actual problem is: in the first place we have to control the first term (3.5) which is suppressed by $\tilde{\rho}_{\mathrm{fin}}(s,s')/\tilde{\rho}_{\mathrm{ini}}(s,s')$ and in regions where $\sigma^{(0)}(s)/\sigma^{(0)}(s')$ is small but otherwise is of $O(1)$.

A measure for the relative importance of the disturbing model-dependent FSR term $\delta^{\mathrm{r.r.}}$ is

$$\delta r^{\mathrm{FSR}}(s',s) = \frac{\tilde{\sigma}^{(\gamma)}(s') - \sigma^{(0)}(s')}{\sigma^{(0)}(s')}, \quad (3.7)$$

which is indeed a measure for the importance of FSR as given by the radiator function $\tilde{\rho}_{\mathrm{fin}}(s,s')$. It has to be compared with (2.13) which measures the FSR in the integrated form (2.5) for the case of a scan experiment. Obviously, (3.7) definitely does not account for the $(\alpha/\pi)\eta(s')$ term in $\sigma^{(\gamma)}(s')$. We may consider it, however, as a measure for the model dependence of the extraction of the undressed $\sigma^{(0)}(s')$. Thus let us point out once more that it would be misleading to think that $\tilde{\sigma}^{(\gamma)}(s')$ in any sense would approximate $\sigma^{(\gamma)}(s')$ to better than the $O(1)$ level. Although it includes FSR effects, it does not include the ones we are looking for.

The size of FSR effects for the radiative return scenario is depicted in Fig. 7.

Clearly the effect is large both in the soft ($s' \lesssim s$) and the hard ($s' \ll s$) photon limits. It is small only where it is suppressed by a large $\sigma^{(0)}(s')$, i.e., around the ρ resonance.

In fact, the increase of $\delta r^{\mathrm{FSR}}(s',s)$ in the soft photon regime (for $s' \to s$) just means that soft photon FSR effects become large and does not imply a large model dependence. Hence taking (3.7) as an estimation of the model error would be too rough. It does not yet take into account that sQED describes well the soft photon regime. In order to get a more realistic measure for the model dependence we have to proceed as in the previous section.

Given the experimental distribution $(\mathrm{d}\sigma/\mathrm{d}s')_{\mathrm{obs}}$ in (3.2) the extracted $\sigma^{(0)}(s')$ depends on the unknown FSR radiator $\tilde{\rho}_{\mathrm{fin}}(s,s')$, which we again model by sQED. Analogously to our analysis for the scan scenario [see (2.14)] we compare the result for a scalar versus a fermionic radiator by looking at

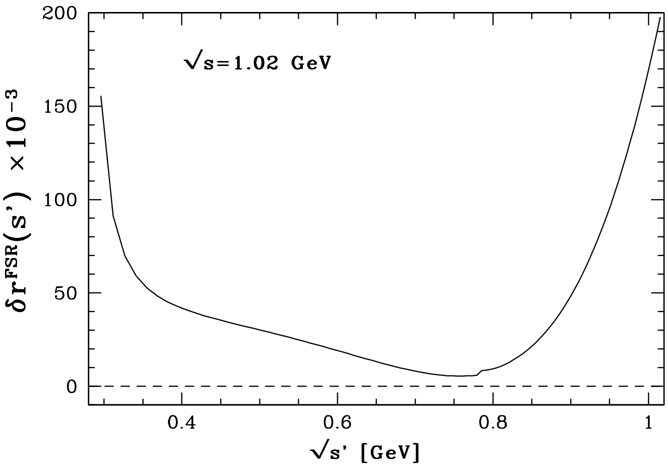

Fig. 7. Importance of FSR for sQED. The curve shows the scenario for the extraction of $|F_\pi^{(0)}(s')|^2$ in radiative return experiments ($s^{1/2} = M_\Phi$) estimated by once including and once excluding final state corrections [see (3.7)]

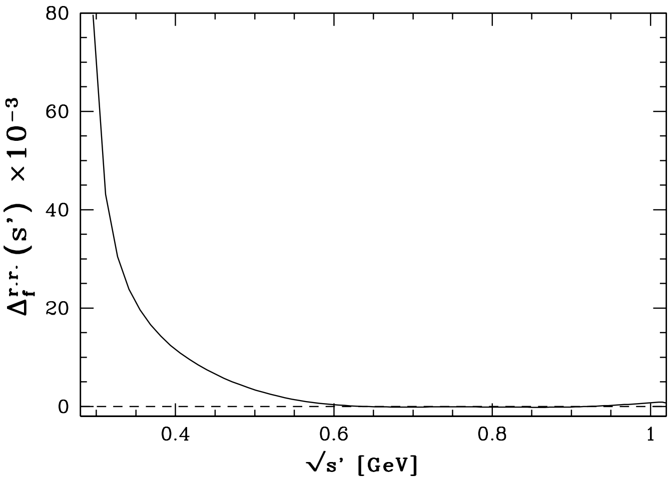

Fig. 8. Estimated relative model error for the extraction of $|F_\pi^{(0)}(s')|^2$ in radiative return experiments ($s^{1/2} = M_\Phi$). The curve describes a scenario where the model error is estimated by a comparison of FSR once from scalar particles and once from fermions of the same charge and mass

$$\Delta_{\mathrm{f}}^{\mathrm{r.r.}}(s') = \frac{\sigma^{(0),\mathrm{f}}(s') - \sigma^{(0)}(s')}{\sigma^{(0)}(s')}, \tag{3.8}$$

with $\sigma^{(0)}(s')$ extracted assuming sQED and $\sigma^{(0),\mathrm{f}}(s')$ extracted assuming a fermionic radiator $\tilde{\rho}^{\mathrm{f}}_{\mathrm{fin}}(s,s')$ [see (B.2) and (B.3)].

The result is shown in Fig. 8. We would like to stress once more that (3.2) does not incorporate the ISR $\otimes$ FSR and IFS effects, which account for an additional $O(\alpha)$ model error contribution. Thus, Fig. 8 when taken without this proviso is misleading at energies where the $O(1)$ term is kinematically suppressed to be smaller than the missing, presently unknown, $O(\alpha)$ terms[11], which are expected to be at the few per mill level.

Clearly, the "scalar versus fermionic" scenario gives, as expected, a small model dependence for the soft photon region but the model error remains large for pion pair production near threshold energies where hard photons are involved necessarily.

We repeat that for radiative return experiments we cannot see a possibility to measure the FSR-inclusive cross section $\sigma^{(\gamma)}(s)$, at least not in some obvious way. We only are able to extract the undressed cross section $\sigma^{(0)}(s)$. At $O(\alpha)$ precision even this is only possible in a model-dependent way, since the problems are the same as the ones we have addressed earlier for the exclusive scan measurements. To get $\sigma^{(\gamma)}(s)$ we must add the FSR as given by sQED, with the drawback that we have to live with the model dependence as illustrated by Fig. 8. Since the disturbing second term in (3.2) is much smaller for a radiative return experiment at a B-factory, it seems that the chances to get a model-independent determination of $\sigma^{(0)}(s')$ there could be good for what concerns the theoretical uncertainties associated with FSR. It can be easily checked that for such measurements at a B-factory indeed the model error is limited by higher order FSR effects (not considered here) since the $O(1)$ FSR contribution is essentially 0 due to the fact that $\sigma_{\pi\pi}(s = M^2_{\Upsilon_{4S}})/\sigma_{\pi\pi}(s \leq 1\,\mathrm{GeV}) \lesssim 0.04$. The observed cross section, on the other hand, is reduced by about two orders of magnitude with respect to Φ-factory measurements. However, for the BABAR experiment at SLAC this drawback is compensated by a very high luminosity which is about a factor of 400 larger than at the DAΦNE collider. It is important to note that a good control of ISR will be required since the gap between s' and s is much larger than for Φ-factories. In particular the singlet initial state pair production channel "$e^+e^- \to \pi^+\pi^-e^+e^-$" yields the dominant contribution to the inclusive channel "$e^+e^- \to \pi^+\pi^- +$ anything" at B-factory energies, being about a factor of 3 larger around the ρ peak and even a factor of about 30 larger near $\pi\pi$ threshold than the contribution from "$e^+e^- \to \pi^+\pi^- +$ photons". Higher order photonic corrections also have to be taken into account. Leading log photonic $O(\alpha^3)$ corrections here contribute about 0.3 per cent to the inclusive spectral function.

Let us now apply C-symmetric angular cuts to the calculations concerning the radiative return method. Cut objects have been defined in (B.18)–(B.21).

Analogously to the case without cuts in (3.7) the importance of the FSR contribution for a C-symmetric cut scenario can be defined as the relative deviation of

$$\tilde{\sigma}^{(\gamma)}_{\mathrm{cut}}(s') = \frac{1}{\rho^{\mathrm{cut}}_{\mathrm{ini}}(s',s)} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)_{\mathrm{obs,cut}} \tag{3.9}$$

from the undressed cross section $\sigma^{(0)}_{\mathrm{cut}}$:

$$\delta r^{\mathrm{FSR}}_{\mathrm{cut}}(s') = \frac{\tilde{\sigma}^{(\gamma)}_{\mathrm{cut}}(s') - \sigma^{(0)}_{\mathrm{cut}}(s')}{\sigma^{(0)}_{\mathrm{cut}}(s')}. \tag{3.10}$$

[11] Their evaluation would require a full two-loop calculation of the process $e^+e^- \to \pi^+\pi^-$

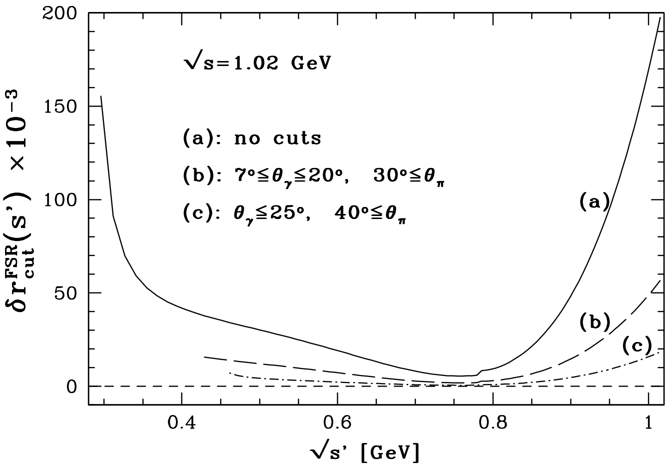

Fig. 9. Relative FSR contribution as given by sQED obscuring the extraction of $|F_\pi^{(0)}(s')|^2$ in radiative return experiments ($s^{1/2} = M_\Phi$) for different angular cuts [see (3.10)]. Curve (a) is the same as the curve in Fig. 7. Curve (b) corresponds to the cut scenario where only events are taken into account for which the laboratory angle between the pion momenta and the beam axis θ_π is larger than 30° and the laboratory photon angle θ_γ is restricted to a region $7^\circ \leq \theta_\gamma \leq 20^\circ$. In a similar way curve (c) corresponds to $\theta_\pi \geq 40^\circ$ and $\theta_\gamma \leq 25^\circ$

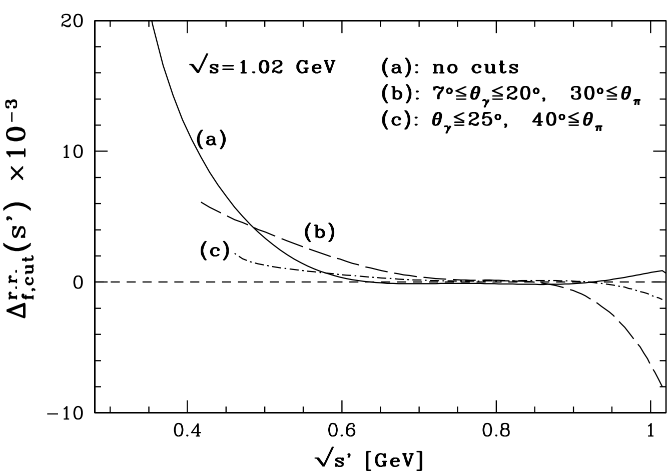

Fig. 10. Estimated relative model error in per mill for the extraction of $|F_\pi^{(0)}(s')|^2$ in radiative return experiments ($s^{1/2} = M_\Phi$) for different angular cuts. Here the scenario is shown where the model error is estimated by a comparison of FSR from scalar particles with FSR from fermions of the same charge and mass [see (3.13)]. The cut scenarios (a), (b) and (c) are the same as in Fig. 9

Again we may "guesstimate" a model uncertainty by replacing and comparing the sQED model with the fQED model. In the latter case we replace $\rho_{\rm fin}^{\rm cut}(s,s')$ by

$$\rho_{\rm fin,f}^{\rm cut}(s,s') = \frac{1}{\sigma_{0,\rm cut}^{\rm point,f}(s)} \left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)_{\rm fin,cut}^{\rm point,f} \qquad (3.11)$$

where

$$\sigma_{0,\rm cut}^{\rm point,f}(s) = \pi \frac{\alpha^2 \beta_\pi^3 \cos\Theta_\pi^{\rm M}}{s}$$

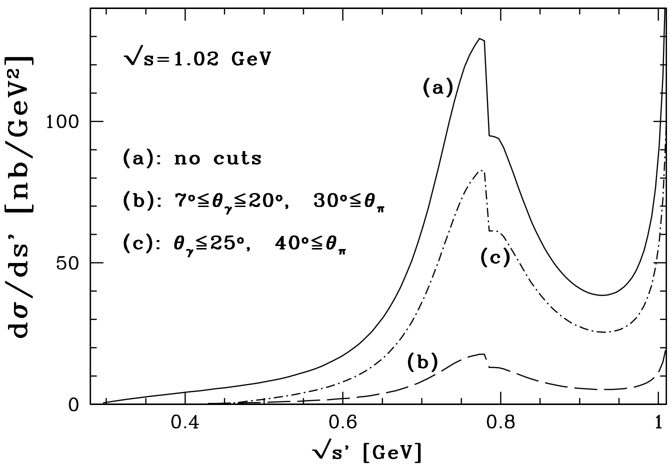

Fig. 11. Pion pair invariant mass distribution in radiative return experiments ($s^{1/2} = M_\Phi$) for the three discussed cut scenarios

$$\times \left(\frac{s+4m_\pi^2}{s-4m_\pi^2} + \frac{1}{3}\cos^2\Theta_\pi^{\rm M}\right), \qquad (3.12)$$

$\Theta_\pi^{\rm M}$ being the minimal angle between the pion momenta and the beam axis allowed by the given cuts. In analogy to the case without cuts in (3.8) we then may define a model error as

$$\Delta_{\rm f,cut}^{\rm r.r.}(s') = \frac{\sigma_{\rm cut}^{(0),\rm f}(s') - \sigma_{\rm cut}^{(0)}(s')}{\sigma_{\rm cut}^{(0)}(s')}. \qquad (3.13)$$

Figure 9 shows the FSR contribution and Fig. 10 the estimated model error for different kinematic cuts. It is interesting to note that the FSR contribution can be clearly reduced by the chosen kinematic cuts. This is especially obvious in the soft photon region. The suppression of the model error, being estimated by the scalar versus fermionic scenario, by the considered cuts, however, is not obvious. As a matter of fact the cross sections drop out very quickly for low ${s'}^{1/2}$ and vanish (up to higher order effects) below ${s'}^{1/2} \simeq 0.5\,\text{GeV}$ (see Fig. 11).

Finally we would like to comment on the estimation of FSR from a measurement of the pion forward–backward asymmetry [15], defined as

$$A_{\rm FB}(s') = \frac{\left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)^{\cos\theta_\pi<0} - \left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)^{\cos\theta_\pi>0}}{\left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)^{\cos\theta_\pi<0} + \left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)^{\cos\theta_\pi>0}}$$

$$= \frac{\left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)_{\rm int}^{\cos\theta_\pi<0}}{\left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)_{\rm ini}^{\cos\theta_\pi<0} + \left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)_{\rm fin}^{\cos\theta_\pi<0}}. \qquad (3.14)$$

Here θ_π is the angle between the momentum of the incoming e^- and the outgoing π^-. One could think that if $A_{\rm FB}$ is small also the FSR contribution must be suppressed, because, as can be seen from (3.14), the IFS contribution to the spectral function $({\rm d}\sigma/{\rm d}s')_{\rm int}$ determines

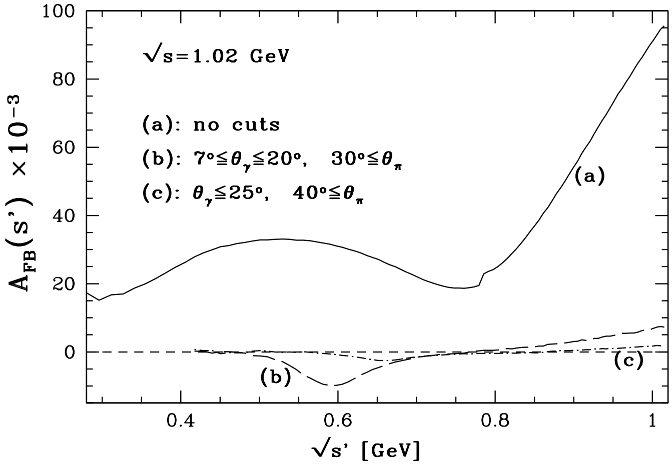

Fig. 12. Forward–backward asymmetry in per mill of negatively charged pions in radiative return experiments. θ_π is here the angle between the momentum of the incoming e^- and the outgoing π^-

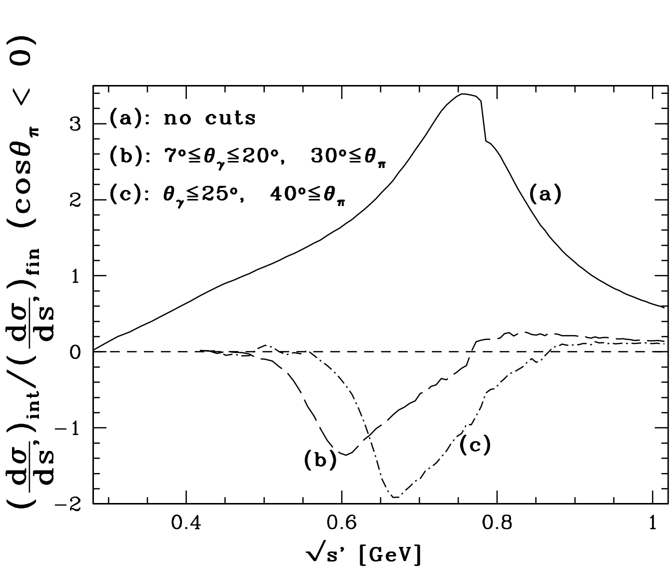

Fig. 13. Ratio of the IFS contribution to final state contribution for $\cos\theta_\pi < 0$. Here θ_π is the angle between the momentum of the incoming e^- and the outgoing π^-

the size of $A_{\rm FB}$. The smallness of $A_{\rm FB}$, enhanced by kinematic cuts, could then be taken as a measure for the suppression of FSR in experiments. In Fig. 12 $A_{\rm FB}$ is shown for the three different cut scenarios. If we compare Fig. 12 with Figs. 7, 8, 9 and 10 we observe that we would get a different estimation of the model dependence from $A_{\rm FB}$ than from our previous analysis of FSR. For example, for the case of no cuts we get an increasing model dependence from our investigation of FSR if s' approaches threshold while $A_{\rm FB}$ decreases for $s' \to 4m_\pi^2$.

Obviously, $A_{\rm FB}$ is affected by FSR and hence can only be predicted by assuming a model like sQED. A comparison of such a prediction with the experimental data is able to shed light on the validity of such a model. For instance, if we observe a good agreement of the measured $A_{\rm FB}$ with the sQED prediction we could expect that likely also the FSR prediction by sQED could be a good approximation.

Of course $A_{\rm FB}$ is not a direct measure of FSR and hence of the model dependence related to it. In fact the ratio of the FSR contribution and the IFS contribution is a strongly varying function of s' (see Fig. 13). While in the region of the ρ resonance this ratio is relatively large it becomes small for low s'. This is true no matter whether cuts are applied or not.

4 Conclusions

The importance of FSR for the extraction of the pion form factor from experimental data and for determining $a_\mu^{\rm had}$ has been discussed both for scan and radiative return experiments.

We have shown that, by a photon-inclusive measurement and just subtracting ISR, a direct extraction of the FSR-inclusive cross section $\sigma^{(\gamma)}$ in scan experiments is possible with excellent accuracy. In this case the model error is due to the breaking of ISR$\otimes$FSR factorization by kinematic cuts and is of the order of a few per mill for the discussed C-symmetric angular cuts.

On the other hand for exclusive measurements it is much more difficult to give a reliable estimation of the model error. The main reason is that without relying on ad hoc models like sQED we are not able to disentangle the QCD quantity (pion form factor) to be extracted from the real and virtual QED corrections.

For radiative return measurements a direct extraction of $\sigma^{(\gamma)}(s)$ is not possible. In fact we can neither obtain $\sigma^{(\gamma)}(s)$ nor $\sigma^{(0)}(s)$ at $O(\alpha)$ FSR precision without resorting to a model. Furthermore we have to deal with an $O(1)$ FSR background which is under control only if one of the following criteria apply:

(i) it can be subtracted by using factorization in the soft photon region ($s' \simeq s$),
(ii) it is suppressed by kinematic cuts (where it is possible) or
(iii) it is negligible in regions where $\sigma^{(0)}(s') \gg \sigma^{(0)}(s)$. We have shown that at ϕ factories like DAΦNEwe can control this FSR background only above ${s'}^{1/2} \simeq 500\,{\rm MeV}$.

We also had a look at the pion forward–backward asymmetry which is a model-dependent quantity and thus is able to test model predictions against reality. Suppose that the data would agree well with the sQED prediction; then this could be an indication that sQED also is able to describe FSR to some extent. However, it is not possible to estimate the FSR contribution in a straightforward way from a measurement of $A_{\rm FB}$. The ratio of the FSR and the IFS correction is a strongly varying function of ${s'}^{1/2}$. $A_{\rm FB}$ becomes small for low ${s'}^{1/2}$ although the final state correction and hence the related model dependence is large.

For radiative return measurements of $|F_\pi^{(0)}(s)|^2$ at B-factories the $O(1)$ FSR background term is practically absent. The model uncertainty due to FSR for the extraction of the undressed pion form factor in this case is determined by higher order effects. At BABAR the smallness of the observed cross section is compensated by a high luminosity. Initial state corrections here have to be known to a high precision. In particular for $s'^{1/2} \leq 1\,\mathrm{GeV}$ the dominant pion pair production channel is $e^+e^- \to \pi^+\pi^-e^+e^-$.

In conclusion, to measure the FSR-inclusive pion form factor $|F_\pi^{(\gamma)}(s)|^2$ precisely and in a model-independent manner at low energies, precise data from photon-inclusive scan experiments will be indispensable. We hope that such an experiment will be possible at DAΦNEat a later stage.

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Appendix

A Factorization of ISR and FSR

Consider the process "$e^+e^- \to \gamma^* \to X +$ photons" where X is an arbitrary non-photonic final state and the photons can either be emitted from the initial state (IS) or the final state (FS). We are interested in an expression for the inclusive total cross section $\sigma_{\mathrm{incl}} \equiv \sigma(e^+e^- \to X +$ photons) in terms of the FSR-inclusive cross section that is dressed by all real and virtual FS photonic corrections $\sigma^{(\gamma)}$ and a universal IS radiator function $\rho_{\mathrm{ini}}^{\mathrm{incl}}$ corresponding to all IS real and virtual corrections. As will be shown in the following such a factorization is in fact possible up to $O(\alpha^2)$ IFS real and virtual QED corrections.

Let us consider first the process $e^+e^- \to \gamma^* \to X + r\gamma$, where r is a given number of real photons which can be emitted either from the IS or the FS. The amplitude $\mathcal{M}^{(r)}$ corresponding to this process can be written as the sum of all sub-amplitudes $\mathcal{M}^{(v_\mathrm{i},v_\mathrm{f})}_{(r_\mathrm{i},r_\mathrm{f})}$ corresponding to r_i real and v_i virtual photons attached to the IS e^+e^- pair, r_f real and v_f virtual photons attached to the final state X, and v_{int} additional virtual photons connecting the IS and the FS. For given r we have the condition $r_\mathrm{i} + r_\mathrm{f} = r$. In the following we will neglect box-like diagrams; thus, we put $v_{\mathrm{int}} = 0$. Hence we only keep the pure IS and FS virtual corrections (we will come back to the IFS contributions later). The IR divergences of the virtual corrections are assumed to be regularized by a small photon mass. Without such an IR regulator $\mathcal{M}^{(r)}$ would not be defined. Obviously $\mathcal{M}^{(r)}$ by itself does not correspond to a physical observable. However, at the end, after summation of all the contributions corresponding to all sub-amplitudes, we will obtain IR finite, physical quantities and the IR regulator can be removed. The amplitude corresponding to the emission of r real photons can now be written as

$$\begin{aligned}\mathcal{M}^{(r)} &= \sum_{r_{\mathrm{i,f}}} \sum_{v_{\mathrm{i,f}}=0}^{\infty} \mathcal{M}^{(v_\mathrm{i},v_\mathrm{f})}_{(r_\mathrm{i},r_\mathrm{f})}\Big|_{r_\mathrm{i}+r_\mathrm{f}=r} \\ &= \sum_{r_{\mathrm{i,f}}} \sum_{v_{\mathrm{i,f}}=0}^{\infty} \Big[A_\mu^{(r_\mathrm{i},v_\mathrm{i})}(q,q_\mathrm{V},\{k^{(\mathrm{i})}\}) \\ &\quad \times B^\mu_{(r_\mathrm{f},v_\mathrm{f})}(q_\mathrm{V},\{k^{(X)}\},\{k^{(\mathrm{f})}\})\Big]_{r_\mathrm{i}+r_\mathrm{f}=r}, \quad \text{(A.1)}\end{aligned}$$

where $\{k^{(\mathrm{i})}\} = \{k_1^{(\mathrm{i})}\dots k_{r_\mathrm{i}}^{(\mathrm{i})}\}$ are the IS real photon momenta, $\{k^{(\mathrm{f})}\} = \{k_1^{(\mathrm{f})}\dots k_{r_\mathrm{f}}^{(\mathrm{f})}\}$ the FS real photon momenta, $\{k^{(X)}\} = \{k_1^{(X)}\dots k_{n_X}^{(X)}\}$ the momenta of the non-photonic FS particles, $q = p_1 + p_2$ the sum of the incoming e^+e^- momenta and finally $q_\mathrm{V} = q - \sum k^{(\mathrm{i})} = \sum k^{(X)} + \sum k^{(\mathrm{f})}$ the momentum of the virtual photon $\gamma^*(q_\mathrm{V})$ connecting the IS and the FS. In (A.1) we have written the sub-amplitudes $\mathcal{M}^{(v_\mathrm{i},v_\mathrm{f})}_{(r_\mathrm{i},r_\mathrm{f})}$ as contractions of rank-1 tensors $A_\mu^{(r_\mathrm{i},v_\mathrm{i})}$ containing the IS real and virtual corrections with the rank-1 tensors $B^\mu_{(r_\mathrm{f},v_\mathrm{f})}$ containing the FS real and virtual corrections and the photon propagator related to $\gamma^*(q_\mathrm{V})$. This factorization of the amplitude is only possible because we have neglected the virtual IFS contributions. Defining

$$\begin{aligned}&\tilde{A}_\mu^{(r_\mathrm{i})}(q,q_\mathrm{V},\{k^{(\mathrm{i})}\}) \\ &\quad= \sum_{v_\mathrm{i}=0}^{\infty} A_\mu^{(r_\mathrm{i},v_\mathrm{i})}(q,q_\mathrm{V},\{k^{(\mathrm{i})}\}), \\ &\tilde{B}^\mu_{(r_\mathrm{f})}(q_\mathrm{V},\{k^{(X)}\},\{k^{(\mathrm{f})}\}) \\ &\quad= \sum_{v_\mathrm{f}=0}^{\infty} B^\mu_{(r_\mathrm{f},v_\mathrm{f})}(q_\mathrm{V},\{k^{(X)}\},\{k^{(\mathrm{f})}\}), \quad \text{(A.2)}\end{aligned}$$

thus summing over all virtual IS and FS virtual corrections, we can write $\mathcal{M}^{(r)}$ simply as

$$\begin{aligned}\mathcal{M}^{(r)} = \Bigg\{&\Bigg[\sum_{r_\mathrm{i}} \tilde{A}_\mu^{(r_\mathrm{i})}(q,q_\mathrm{V},k^{(\mathrm{i})})\Bigg] \\ &\times \Bigg[\sum_{r_\mathrm{f}} \tilde{B}^\mu_{(r_\mathrm{f})}(q_\mathrm{V},k^{(X)},k^{(\mathrm{f})})\Bigg]\Bigg\}_{r_\mathrm{i}+r_\mathrm{f}=r}. \quad \text{(A.3)}\end{aligned}$$

Let us remember that $\mathcal{M}^{(r)}$ contains all IS and FS virtual corrections to the given channel with r real (IS or FS) photons. Squaring the amplitude, averaging over the incoming e^+e^- spins and summing over the spins of the ISR photons $s_{(\mathrm{i})}$, the spins of the FSR photons $s_{(\mathrm{f})}$ and the spins of the non-photonic particles $s_{(X)}$ yields

$$\begin{aligned}\overline{|\mathcal{M}^{(r)}|^2} &= \frac{1}{4}\sum_{s_{(X)}}\sum_{s_{(\mathrm{i})}}\sum_{s_{(\mathrm{f})}}\sum_{r_\mathrm{i},r_\mathrm{f}} |\mathcal{M}^{(r_\mathrm{i},r_\mathrm{f})}|^2 + \text{IFS terms} \\ &= \sum_{r_\mathrm{i},r_\mathrm{f}} \overline{|\mathcal{M}^{(r_\mathrm{i},r_\mathrm{f})}|^2} + \text{IFS terms}, \quad \text{(A.4)}\end{aligned}$$

where "IFS terms" corresponds to real photon IFS contributions. Since we neglected already the virtual IFS contributions we have to do the same for the real IFS contributions. Otherwise we would have no cancellation of the IR divergences if later, after summing up all contributions $r = 0, \dots, \infty$, we want to remove the IR regulator. Neglecting the IFS contributions we can now express the squared amplitude $\overline{|\mathcal{M}^{(r)}|^2}$ as a sum of the incoherent contributions $\overline{|\mathcal{M}^{(r_\mathrm{i},r_\mathrm{f})}|^2}$ which can be written as a contraction of an IS and a FS tensor, respectively:

$$\overline{|\mathcal{M}^{(r_\mathrm{i},r_\mathrm{f})}|^2} = E^{(r_\mathrm{i})}_{\mu\nu}(q, q_\mathrm{V}, \{k^{(\mathrm{i})}\}) F^{\mu\nu}_{(r_\mathrm{f})}(q_\mathrm{V}, \{k^{(X)}\}, \{k^{(\mathrm{f})}\}), \tag{A.5}$$

with

$$E^{(r_\mathrm{i})}_{\mu\nu}(q, q_\mathrm{V}, \{k^{(\mathrm{i})}\}) = \frac{1}{4}\sum_{s_{(\mathrm{i})}} \tilde{A}^{(r_\mathrm{i})}_{\mu}(q, q_\mathrm{V}, \{k^{(\mathrm{i})}\}) \tilde{A}^{(r_\mathrm{i})*}_{\nu}(q, q_\mathrm{V}, \{k^{(\mathrm{i})}\}), \tag{A.6}$$

$$F^{\mu\nu}_{(r_\mathrm{f})}(q_\mathrm{V}, \{k^{(X)}\}, \{k^{(\mathrm{f})}\}) = \sum_{s_{(\mathrm{f})}}\sum_{s_{(X)}} \tilde{B}^{\mu}_{(r_\mathrm{f})}(q_\mathrm{V}, \{k^{(X)}\}, \{k^{(\mathrm{f})}\}) \times \tilde{B}^{*\nu}_{(r_\mathrm{f})}(q_\mathrm{V}, \{k^{(X)}\}, \{k^{(\mathrm{f})}\}). \tag{A.7}$$

Note that $E^{(r_\mathrm{i})}_{\mu\nu}(q, q_\mathrm{V}, \{k^{(\mathrm{i})}\})$ now only contains the IS real and virtual corrections while $F^{\mu\nu}_{(r_\mathrm{f})}$ only contains the FS real and virtual corrections. Thus, already at this stage we obtain complete factorization of the IS and the FS corrections. Phase space integration over the $r_\mathrm{i} + r_\mathrm{f} + n_X$ particle final state yields the total cross section related to the emission of r_i IS and r_f FS photons [$\beta_e = (1 - 4m_e^2/s)^{1/2}$, $s = q^2$],

$$\sigma_{r_\mathrm{i},r_\mathrm{f}}(s) = \frac{1}{2s\beta_e}\int \prod_{a=1}^{r_\mathrm{i}}\prod_{b=1}^{r_\mathrm{f}}\prod_{c=1}^{n_X} \mathrm{dLips}^{(\mathrm{i})}_a \mathrm{dLips}^{(\mathrm{f})}_b \mathrm{dLips}^{(X)}_c \times (2\pi)^4 \delta^{(4)}\left(q - \sum_{a=1}^{r_\mathrm{i}} k^{(\mathrm{i})}_a - \sum_{b=1}^{r_\mathrm{f}} k^{(\mathrm{f})}_b - \sum_{c=1}^{n_X} k^{(X)}_c\right) \times \overline{|\mathcal{M}^{(r_\mathrm{i},r_\mathrm{f})}|^2}, \tag{A.8}$$

with

$$\mathrm{dLips}^{(\mathrm{i})}_a = \frac{\mathrm{d}^3 k^{(\mathrm{i})}_a}{(2\pi)^3 2E^{(\mathrm{i})}_a}, \quad \mathrm{dLips}^{(\mathrm{f})}_b = \frac{\mathrm{d}^3 k^{(\mathrm{f})}_b}{(2\pi)^3 2E_b}, \quad \mathrm{dLips}^{(X)}_c = \frac{\mathrm{d}^3 k^{(X)}_c}{(2\pi)^3 2E^{(X)}_c}. \tag{A.9}$$

For the real photons the same IR regulator (photon mass) has to be used as for the virtual photons. Hence we are all the time dealing with IR-regularized expressions. Although (A.8) includes now virtual and real photon IS and FS corrections we cannot remove the IR regulator since we included only r real photons but virtual corrections to all orders. $\sigma_{r_\mathrm{i},r_\mathrm{f}}(s)$ is obviously not a physical quantity by itself. Inserting the following identities (with $s_\mathrm{V} = q_\mathrm{V}^2$),

$$1 = \int \mathrm{d}^4 q_\mathrm{V} \delta^{(4)}\left(q_\mathrm{V} - \sum_{b=1}^{r_\mathrm{f}} k^{(\mathrm{f})}_b - \sum_{c=1}^{n_X} k^{(X)}_c\right)$$

and

$$1 = \int \mathrm{d}s_\mathrm{V} \delta(s_\mathrm{V} - q_\mathrm{V}^2), \tag{A.10}$$

into (A.8) yields

$$\sigma_{r_\mathrm{i},r_\mathrm{f}}(s) = \frac{1}{2s\beta_e}\int \mathrm{d}s_\mathrm{V} \int \prod_{a=1}^{r_\mathrm{i}} \mathrm{dLips}^{(\mathrm{i})}_a \frac{\mathrm{d}^3 q_\mathrm{V}}{2q^0_\mathrm{V}} \times \delta^{(4)}\left(q - q_\mathrm{V} - \sum_{a=1}^{r_\mathrm{i}} k^{(\mathrm{i})}_a\right) \times E^{(r_\mathrm{i})}_{\mu\nu}(q, q_\mathrm{V}, \{k^{(\mathrm{i})}\}) \mathcal{F}^{\mu\nu}_{(r_\mathrm{f})}(q_\mathrm{V}), \tag{A.11}$$

with the integrated FSR tensor

$$\mathcal{F}^{\mu\nu}_{(r_\mathrm{f})}(q_\mathrm{V}) = \int \prod_{b=1}^{r_\mathrm{f}}\prod_{c=1}^{n_X} \mathrm{dLips}^{(\mathrm{f})}_b \mathrm{dLips}^{(X)}_c \times (2\pi)^4 \delta^{(4)}\left(q_\mathrm{V} - \sum_{b=1}^{r_\mathrm{f}} k^{(\mathrm{f})}_b - \sum_{c=1}^{n_X} k^{(X)}_c\right) \times F^{\mu\nu}_{(r_\mathrm{f})}(q_\mathrm{V}, \{k^{(X)}\}, \{k^{(\mathrm{f})}\}). \tag{A.12}$$

From Lorentz covariance follows that the integrated FSR tensor can be written as a linear combination of the two linear independent tensors $g^{\mu\nu}$ and $q^\mu_\mathrm{V} q^\nu_\mathrm{V}$:

$$\mathcal{F}^{\mu\nu}_{(r_\mathrm{f})}(q_\mathrm{V}) = A^{(r_\mathrm{f})}(s_\mathrm{V}) g^{\mu\nu} + B^{(r_\mathrm{f})}(s_\mathrm{V}) q^\mu_\mathrm{V} q^\nu_\mathrm{V} = A^{(r_\mathrm{f})}(s_\mathrm{V}) \left(g^{\mu\nu} - \frac{q^\mu_\mathrm{V} q^\nu_\mathrm{V}}{s_\mathrm{V}}\right) = \frac{1}{3}\mathrm{tr}\left[\mathcal{F}^{\mu}_{(r_\mathrm{f})\nu}(q_\mathrm{V})\right]\left(g^{\mu\nu} - \frac{q^\mu_\mathrm{V} q^\nu_\mathrm{V}}{s_\mathrm{V}}\right). \tag{A.13}$$

For the second equality in (A.13) gauge invariance has been used, implying the Ward identity $q_{\mathrm{V}\mu}\mathcal{F}^{\mu\nu}_{(r_\mathrm{f})}(q_\mathrm{V}) = 0$. Note that for the special case of no ISR photon emission ($r_\mathrm{i} = 0$) the related total cross section corresponding to the emission of $r_\mathrm{f} = r$ FSR photons can be written as

$$\sigma_{0,r_\mathrm{f}}(s) = \frac{1}{2s\beta_e} E^{(0)}_{\mu\nu}(q) \mathcal{F}^{\mu\nu}_{(r_\mathrm{f})}(q), \tag{A.14}$$

with the lowest order IS tensor

$$E^{(0)}_{\mu\nu}(q) = e^2\left(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - \frac{s}{2}g^{\mu\nu}\right). \tag{A.15}$$

Taking into account the Ward identity $q^\mu E^{(0)}_{\mu\nu}(q) = 0$ and using (A.13) we can write

$$\sigma_{0,r_\mathrm{f}}(s) = \frac{1}{2s\beta_e(s)}\mathrm{tr}[E^{(0)\nu}_{\mu}(q)]\frac{1}{3}\mathrm{tr}\left[\mathcal{F}^{\mu}_{(r_\mathrm{f})\nu}(q)\right] = -\frac{e^2}{3}\frac{s + 2m_e^2}{2s\beta_e(s)}\mathrm{tr}\left[\mathcal{F}^{\mu}_{(r_\mathrm{f})\nu}(q)\right]. \tag{A.16}$$

For the general case including r_i ISR photons the following Ward identity holds:

$$q^\mu_\mathrm{V} E^{(r_\mathrm{i})}_{\mu\nu}(q, q_\mathrm{V}, \{k^{(\mathrm{i})}\}) = 0. \tag{A.17}$$

Inserting the expression for $\mathcal{F}^{\mu\nu}(q_\mathrm{V})$ in (A.13) into (A.11) we can write the cross section corresponding to the emission of r_i ISR and r_f FSR photons as

$$\sigma_{r_\mathrm{i},r_\mathrm{f}}(s) = \frac{1}{2s\beta_e}\int \mathrm{d}s_\mathrm{V}\frac{1}{3}\mathrm{tr}\left[\mathcal{F}^{\mu}_{(r_\mathrm{f})\nu}(q_\mathrm{V})\right] \times \int\prod_{a=1}^{r_\mathrm{i}}\mathrm{dLips}^{(\mathrm{i})}_a\frac{\mathrm{d}^3q_\mathrm{V}}{2q^0_\mathrm{V}}\delta^{(4)}\left(q-q_\mathrm{V}-\sum_{a=1}^{r_\mathrm{i}}k^{(\mathrm{i})}_a\right) \times \mathrm{tr}[E^{(r_\mathrm{i})\nu}_\mu(q,q_\mathrm{V},\{k^{(\mathrm{i})}\})] \quad (A.18)$$

(note that $\mathrm{tr}\left[\mathcal{F}^{\mu}_{(r_\mathrm{f})\nu}(q_\mathrm{V})\right] = 3A^{(r_\mathrm{f})}(s_\mathrm{V})$ is only a function of s_V). Hence, using (A.16), we finally arrive at

$$\sigma_{r_\mathrm{i},r_\mathrm{f}}(s) = \int \mathrm{d}s_\mathrm{V}\sigma_{0,r_\mathrm{f}}(s_\mathrm{V})\rho^{(r_\mathrm{i})}_\mathrm{ini}(s,s_\mathrm{V}), \quad (A.19)$$

with

$$\rho^{(r_\mathrm{i})}_\mathrm{ini}(s,s_\mathrm{V}) = -\frac{1}{e^2}\frac{1}{s+2m_e^2}\int\prod_{a=1}^{r_\mathrm{i}}\mathrm{dLips}^{(\mathrm{i})}_a\frac{\mathrm{d}^3q_\mathrm{V}}{2q^0_\mathrm{V}} \times \delta^{(4)}\left(q-q_\mathrm{V}-\sum_{a=1}^{r_\mathrm{i}}k^{(\mathrm{i})}_a\right) \times \mathrm{tr}[E^{(r_\mathrm{i})\nu}_\mu(q,q_\mathrm{V},\{k^{(\mathrm{i})}\})]. \quad (A.20)$$

[Note that for the case of no ISR photons the above equation directly gives $\rho^{(0)}_\mathrm{ini}(s,s_\mathrm{V}) = \delta(s-s_\mathrm{V})$ which just projects out $\sigma_{0,r_\mathrm{f}}(s)$ in (A.19)]. At this point we can ask the question what will be the expression for the inclusive total cross section $\sigma_\mathrm{incl}(s)$ with any number of real and virtual IS and FS photons. Neglecting the IFS contributions we can immediately write $\sigma_\mathrm{incl}(s)$ as the incoherent sum of all ISR $\otimes$ FSR contributions:

$$\sigma_\mathrm{incl}(s) = \sum_{r_\mathrm{i},r_\mathrm{f}=0}^{\infty}\sigma_{r_\mathrm{i},r_\mathrm{f}}(s) \quad (A.21)$$
$$= \int \mathrm{d}s_\mathrm{V}\sigma^{(\gamma)}(s_\mathrm{V})\rho^\mathrm{incl}_\mathrm{ini}(s,s_\mathrm{V}) + O(\alpha^2)_\mathrm{IFS},$$

with

$$\sigma^{(\gamma)}(s_\mathrm{V}) = \sum_{r_\mathrm{f}=0}^{\infty}\sigma_{0,r_\mathrm{f}}(s_\mathrm{V}),$$
$$\rho^\mathrm{incl}_\mathrm{ini}(s,s_\mathrm{V}) = \sum_{r_\mathrm{i}=0}^{\infty}\rho^{(r_\mathrm{i})}_\mathrm{ini}(s,s_\mathrm{V}). \quad (A.22)$$

So in (A.21) we finally expressed the completely inclusive total cross section $\sigma_\mathrm{incl}(s) = \sigma(e^+e^- \to X + \text{photons})$ in a factorized form. Note that $\sigma^{(\gamma)}(s_\mathrm{V})$ now contains the real and virtual FS corrections to all orders and $\rho^\mathrm{incl}_\mathrm{ini}(s,s_\mathrm{V})$ contains the real and virtual IS corrections to all orders. $\sigma^{(\gamma)}(s)$ and $\rho^\mathrm{incl}_\mathrm{ini}(s,s_\mathrm{V})$ are separately IR finite and the IR regulator can therefore be removed. This of course has to be the case since $\sigma_\mathrm{incl}(s)$ is [up to $O(\alpha^2)$ IFS effects] a physical observable.

Using the above formulas it is now straightforward to express σ_incl in (A.21) as a perturbation series in α. For simplicity we will show explicitly the expansion only to $O(\alpha)$. For this we write the FSR-inclusive cross section as the expansion

$$\sigma^{(\gamma)}(s) = \sum_{r_\mathrm{f}=0}^{\infty}\sum_{v_\mathrm{f}=0}^{\infty}\sum_{v'_\mathrm{f}=0}^{\infty}\sigma_{(r_\mathrm{f},v_\mathrm{f},v'_\mathrm{f})}(s), \quad (A.23)$$

with

$$\sigma_{(r_\mathrm{f},v_\mathrm{f},v'_\mathrm{f})}(s) = \frac{1}{2s\beta_e(s)}\int\prod_{b=1}^{r_\mathrm{f}}\prod_{c=1}^{n_X}\mathrm{dLips}^{(\mathrm{f})}_b\mathrm{dLips}^{(X)}_c \quad (A.24)$$
$$\times (2\pi)^4\delta^{(4)}\left(q-\sum_{b=1}^{r_\mathrm{f}}k^{(\mathrm{f})}_b-\sum_{c=1}^{n_X}k^{(X)}_c\right)\overline{|\mathcal{M}|}^2_{(r_\mathrm{f},v_\mathrm{f},v'_\mathrm{f})},$$

and

$$\overline{|\mathcal{M}|}^2_{(r_\mathrm{f},v_\mathrm{f},v'_\mathrm{f})} = -\frac{e^2}{3}(s+2m_e^2) \times \sum_{s_{(\mathrm{f})}}\sum_{s_{(X)}}\Big[B_{(r_\mathrm{f},v_\mathrm{f})\mu}(q,\{k^{(X)}\},\{k^{(\mathrm{f})}\}) \times B^{*\mu}_{(r_\mathrm{f},v'_\mathrm{f})}(q,\{k^{(X)}\},\{k^{(\mathrm{f})}\})\Big]. \quad (A.25)$$

Also the inclusive IS radiator we write as an expansion:

$$\rho^\mathrm{incl}_\mathrm{ini}(s,s_\mathrm{V}) = \sum_{r_\mathrm{i}=0}^{\infty}\sum_{v_\mathrm{i}=0}^{\infty}\sum_{v'_\mathrm{i}=0}^{\infty}\rho_{(r_\mathrm{i},v_\mathrm{i},v'_\mathrm{i})}(s,s_\mathrm{V}) \quad (A.26)$$

with

$$\rho_{(r_\mathrm{i},v_\mathrm{i},v'_\mathrm{i})}(s,s_\mathrm{V}) = -\frac{1}{4e^2}\frac{1}{s+2m_e^2}\int\prod_{a=1}^{r_\mathrm{i}}\mathrm{dLips}^{(\mathrm{i})}_a\frac{\mathrm{d}^3q_\mathrm{V}}{2q^0_\mathrm{V}} \times \delta^{(4)}\left(q-q_\mathrm{V}-\sum_{a=1}^{r_\mathrm{i}}k^{(\mathrm{i})}_a\right)A^{(r_\mathrm{i},v_\mathrm{i})}_\mu(q,q_\mathrm{V},\{k^{(\mathrm{i})}\}) \times A^{*\mu}_{(r_\mathrm{i},v'_\mathrm{i})}(q,q_\mathrm{V},\{k^{(\mathrm{i})}\}). \quad (A.27)$$

The inclusive cross section can then be written as

$$\sigma_\mathrm{incl}(s) = \int \mathrm{d}s_\mathrm{V}\Big\{\Big[\sigma_{(0,0,0)}(s_\mathrm{V})+\sigma_{(1,0,0)}(s_\mathrm{V}) + \sigma_{(0,1,0)}(s_\mathrm{V})+\sigma_{(0,0,1)}(s_\mathrm{V})\Big] \times \Big[\rho_{(0,0,0)}(s,s_\mathrm{V})+\rho_{(1,0,0)}(s,s_\mathrm{V}) + \rho_{(0,1,0)}(s,s_\mathrm{V})+\rho_{(0,0,1)}(s,s_\mathrm{V})\Big]\Big\} + O(\alpha^2)$$
$$= \sigma^{(\gamma)}(s)\left[1+\delta_\mathrm{ini}(s,\Lambda)\right] \quad (A.28)$$
$$+ \int_{s_\mathrm{V}^\mathrm{min}}^{s-2\sqrt{s}\Lambda}\mathrm{d}s_\mathrm{V}\sigma^{(\gamma)}(s_\mathrm{V})\rho_\mathrm{ini}(s,s_\mathrm{V}) + O(\alpha^2),$$

with

$$\begin{aligned}\sigma^{(\gamma)}(s_V) &= \sigma_{(0,0,0)}(s_V) + \sigma_{(1,0,0)}(s_V) \\ &\quad + \sigma_{(0,1,0)}(s_V) + \sigma_{(0,0,1)}(s_V) + O(\alpha^2), \\ \rho_{\rm ini}(s,s_V) &= \rho_{(1,0,0)}(s,s_V), \\ \delta_{\rm ini}(s,\Lambda) &= \int_{s_V^{\rm min}}^{s} {\rm d}s_V \left[\rho_{(0,1,0)}(s,s_V) + \rho_{(0,0,1)}(s,s_V)\right] \\ &\quad + \int_{s-2\sqrt{s}\Lambda}^{s} {\rm d}s_V \rho_{(1,0,0)}(s,s_V). \qquad (A.29)\end{aligned}$$

Here Λ is the soft photon cut off, $\sigma_{(0,0,0)}(s) = \sigma^{(0)}(s)$, and $\rho_{(0,0,0)}(s,s_V) = \delta(s - s_V)$.

The above derivation had many steps; however, it should be stressed that the most important one to get ISR $\otimes$ FSR factorization was the use of Lorentz covariance and gauge invariance in (A.13). If kinematical cuts are applied then ISR$\otimes$FSR factorization breaks down because (A.13) will not be valid any more. Perturbatively this occurs already at $O(\alpha)$, as we show in Appendix B.

B Scalar and fermionic final state corrections to $\pi^+\pi^-$ production

Taking final state corrections up to $O(\alpha)$ into account the observed pion pair invariant mass distribution can be written as the sum of an ISR and an FSR contribution:

$$\left(\frac{{\rm d}\sigma}{{\rm d}s'}\right)_{\rm obs} = \sigma^{(0)}(s')\tilde{\rho}_{\rm ini}(s,s') + \sigma^{(0)}(s)\tilde{\rho}_{\rm fin}(s,s'). \qquad (B.1)$$

For the analytic expression of the initial state radiator function $\tilde{\rho}_{\rm ini}(s,s')$, including radiative corrections up to leading log $O(\alpha^3)$ and leading initial state e^+e^- pair production contributions, we refer to (17) in [13]. Integrating the spectral function in (B.1) over s' yields the observed total cross section $\sigma_{\rm obs}(s)$.

The $O(\alpha)$ final state radiator functions, corresponding to hard photon radiation from scalar particles $\tilde{\rho}_{\rm fin}(s,s')$ and from fermionic particles $\tilde{\rho}^{\rm f}_{\rm fin}(s,s')$, read ($z = s'/s$)

$$\begin{aligned}\tilde{\rho}_{\rm fin}(s,s') &= \frac{1}{s}\Big\{ -\delta(1-z) + \left[1 + \tilde{\delta}^{\rm V+S}_{\rm fin}(s)\right] \qquad (B.2) \\ &\quad \times B_\pi(s,s')\,[1-z]^{B_\pi(s,s')-1} + \tilde{\delta}^{\rm H}_{\rm fin}(s,s')\Big\}, \\ \tilde{\rho}^{\rm f}_{\rm fin}(s,s') &= \frac{1}{s}\Big\{ -\delta(1-z) + \left[1 + \tilde{\delta}^{\rm V+S}_{{\rm fin},f}(s)\right] \qquad (B.3) \\ &\quad \times B_\pi(s,s')\,[1-z]^{B_\pi(s,s')-1} + \tilde{\delta}^{\rm H}_{{\rm fin},f}(s,s')\Big\},\end{aligned}$$

respectively, with the corresponding hard and soft photon functions

$$\tilde{\delta}^{\rm H}_{\rm fin}(s,s') = \frac{2\alpha}{\pi}(1-z)\frac{\beta_\pi(s')}{\beta^3_\pi(s)}, \qquad (B.4)$$

$$\begin{aligned}\tilde{\delta}^{\rm H}_{{\rm fin},f}(s,s') &= \frac{\alpha}{\pi}\frac{(1-z)s}{s+2m_\pi^2}\frac{\beta_\pi(s')}{\beta_\pi(s)} \\ &\quad \times \left[-1 + \frac{1}{\beta_\pi(s')}\log\left(\frac{1+\beta_\pi(s')}{1-\beta_\pi(s')}\right)\right], \qquad (B.5)\end{aligned}$$

$$\begin{aligned}B_\pi(s,s') &= \frac{2\alpha}{\pi}\frac{s'\beta_\pi(s')}{s\beta_\pi(s)} \\ &\quad \times \left[\frac{1+\beta^2_\pi(s')}{2\beta_\pi(s')}\log\left(\frac{1+\beta_\pi(s')}{1-\beta_\pi(s')}\right) - 1\right], \qquad (B.6)\end{aligned}$$

$$\begin{aligned}\tilde{\delta}^{\rm V+S}_{\rm fin}(s) &= \frac{\alpha}{\pi}\Bigg\{\frac{3s-4m_\pi^2}{s\beta_\pi}\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right) \\ &\quad - 2 - 2\log\left(\frac{1-\beta^2_\pi}{4}\right) \\ &\quad - \frac{1+\beta^2_\pi}{2\beta_\pi}\Bigg[\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)\Bigg[\log\left(\frac{1+\beta_\pi}{2}\right) \\ &\quad + \log(\beta_\pi)\Bigg] + \log\left(\frac{1+\beta_\pi}{2\beta_\pi}\right)\log\left(\frac{1-\beta_\pi}{2\beta_\pi}\right) \\ &\quad + 2{\rm Li}_2\left(\frac{2\beta_\pi}{1+\beta_\pi}\right) + 2{\rm Li}_2\left(-\frac{1-\beta_\pi}{2\beta_\pi}\right) \\ &\quad - \frac{2}{3}\pi^2\Bigg]\Bigg\}, \qquad (B.7)\end{aligned}$$

$$\tilde{\delta}^{\rm V+S}_{{\rm fin},f}(s) = \tilde{\delta}^{\rm V+S}_{\rm fin}(s) - \frac{\alpha}{\pi}\frac{1}{2\beta_\pi}\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right). \qquad (B.8)$$

Neglecting soft photon exponentiation, we can write the observed total cross section as the sum of a soft photon contribution and a hard photon contribution:

$$\begin{aligned}\sigma^{(\rm f)}_{\rm obs}(s) &= \sigma^{(0)}(s)\left[1 + \delta_{\rm ini}(s,\Lambda) + \delta^{(\rm f)}_{\rm fin}(s,\Lambda)\right] \\ &\quad + \int_{4m_\pi^2}^{s-2\sqrt{s}\Lambda}{\rm d}s'\rho_{\rm ini}(s,s')\sigma^{(0)}(s') \\ &\quad + \sigma^{(0)}(s)\int_{4m_\pi^2}^{s-2\sqrt{s}\Lambda}{\rm d}s'\rho^{(\rm f)}_{\rm fin}(s,s'), \qquad (B.9)\end{aligned}$$

where

$$\rho_{\rm fin}(s,s') = \frac{1}{s}\left[\tilde{\delta}^{\rm H}_{\rm fin}(s,s') + \frac{B_\pi(s,s')}{1-z}\right], \qquad (B.10)$$

$$\rho^{\rm f}_{\rm fin}(s,s') = \frac{1}{s}\left[\tilde{\delta}^{\rm H}_{{\rm fin},f}(s,s') + \frac{B_\pi(s,s')}{1-z}\right], \qquad (B.11)$$

$$\delta_{\rm fin}(s,\Lambda) = \log\left(\frac{2\Lambda}{\sqrt{s}}\right)B_\pi(s'=s) + \tilde{\delta}^{\rm V+S}_{\rm fin}(s), \qquad (B.12)$$

$$\delta^{\rm f}_{\rm fin}(s,\Lambda) = \log\left(\frac{2\Lambda}{\sqrt{s}}\right)B_\pi(s'=s) + \tilde{\delta}^{\rm V+S}_{{\rm fin},f}(s) \qquad (B.13)$$

Taking now only the leading order contribution to the total cross section and the final state corrections into account leads to the $O(\alpha)$ FSR-inclusive cross section

$$\begin{aligned}\sigma^{(\gamma)(,{\rm f})}(s) &= \sigma^{(0)}(s)\Bigg\{1 + \delta^{(\rm f)}_{\rm fin}(s,\Lambda) \\ &\quad + \int_{4m_\pi^2}^{s-2\sqrt{s}\Lambda}{\rm d}s'\rho^{(\rm f)}_{\rm fin}(s,s')\Bigg\} \\ &= \sigma^{(0)}(s)\left[1 + \frac{\alpha}{\pi}\eta^{(\rm f)}(s)\right]. \qquad (B.14)\end{aligned}$$

The analytic $O(\alpha)$ expression for the integrated final state correction factors read

$$\eta(s) = \frac{1+\beta_\pi^2}{\beta_\pi} \left\{ 4\mathrm{Li}_2\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right) + 2\mathrm{Li}_2\left(-\frac{1-\beta_\pi}{1+\beta_\pi}\right) - 3\log\left(\frac{2}{1+\beta_\pi}\right)\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right) - 2\log(\beta_\pi)\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)\right\} - 3\log\left(\frac{4}{1-\beta_\pi^2}\right) - 4\log(\beta_\pi) + \frac{1}{\beta_\pi^3}\left[\frac{5}{4}(1+\beta_\pi^2)^2 - 2\right]\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right) + \frac{3}{2}\frac{1+\beta_\pi^2}{\beta_\pi^2} \quad \text{(B.15)}$$

for a scalar particle final state and

$$\eta^{\mathrm{f}}(s) = \eta(s) + \frac{1}{s+2m_\pi^2}\frac{1}{2s\beta_\pi} \times \left[(s^2+2m_\pi^4)\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right) - \frac{s\beta_\pi}{2}(5s-2m_\pi^2)\right] + \frac{1}{s^2\beta_\pi^3} \times \left[4m_\pi^2(s-m_\pi^2)\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right) - s\beta_\pi(s+2m_\pi^2)\right] - \frac{1}{2\beta_\pi}\log\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right) \quad \text{(B.16)}$$

for a fermionic final state.

Finally some comments on applying kinematic cuts, leading to a breaking of ISR $\otimes$ FSR factorization (see Appendix A). Treating FSR by sQED we can write the spectral function with C-symmetric kinematic cuts as

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)_{\mathrm{obs,cut}} = |F_\pi^{(0)}(s')|^2 \left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)^{\mathrm{point}}_{\mathrm{ini,cut}} + |F_\pi^{(0)}(s)|^2 \left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)^{\mathrm{point}}_{\mathrm{fin,cut}} + O(\alpha^2), \quad \text{(B.17)}$$

where $(\mathrm{d}\sigma/\mathrm{d}s')^{\mathrm{point}}_{\mathrm{ini,cut}}$ and $(\mathrm{d}\sigma/\mathrm{d}s')^{\mathrm{point}}_{\mathrm{fin,cut}}$ are the corresponding IS and FS spectral function for point-like, scalar particles. Note that within the considered sQED model we treat the non-perturbative QCD effects (pion form factor) in a factorized way which means that the FSR corrections are treated within pure sQED and therefore do not affect the pion form factor. By some straightforward manipulations we can write the total photon-inclusive cross section (2.3) for the given cuts in a similar form as in (A.28).

Let us note that s_{V} in (2.3) is a formal integration variable since in $\sigma^{(\gamma)}_{\mathrm{cut}}(s_{\mathrm{V}})$ s_{V} corresponds to the invariant mass square of the pions including FSR, while in $\rho^{\mathrm{cut}}_{\mathrm{ini}}(s, s_{\mathrm{V}})$ s_{V} corresponds to the invariant mass square s' of the pions excluding FSR.

The cut quantities are defined as follows:

$$\sigma^{(0)}_{\mathrm{cut}}(s) = |F_\pi^{(0)}(s)|^2 \sigma^{0,\mathrm{point}}_{\mathrm{cut}}(s) \quad \text{(B.18)}$$

with

$$\sigma^{0,\mathrm{point}}_{\mathrm{cut}}(s) = \pi\frac{\alpha^2\beta_\pi^3\cos\Theta_\pi^{\mathrm{M}}}{2s}\left(1 - \frac{1}{3}\cos^2\Theta_\pi^{\mathrm{M}}\right). \quad \text{(B.19)}$$

Here Θ_π^{M} is the minimal angle between the pion momenta and the beam axis allowed by the given cuts. Similarly, we obtain

$$\sigma^{(\gamma)}_{\mathrm{cut}}(s) = |F_\pi^{(\gamma)}(s)|^2 \sigma^{0,\mathrm{point}}_{\mathrm{cut}}(s) \quad \text{(B.20)}$$

and

$$\rho^{(\mathrm{cut})}_{\mathrm{ini}}(s,s') = \frac{1}{\sigma^{0,\mathrm{point}}_{(\mathrm{cut})}(s')}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)^{\mathrm{point}}_{\mathrm{ini,(cut)}}, \quad \rho^{(\mathrm{cut})}_{\mathrm{fin}}(s,s') = \frac{1}{\sigma^{0,\mathrm{point}}_{(\mathrm{cut})}(s)}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s'}\right)^{\mathrm{point}}_{\mathrm{fin,(cut)}}. \quad \text{(B.21)}$$

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